Three-magnon problem for exactly rung-dimerized spin ladders: from general outlook to Bethe Ansatze

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Abstract

Three-magnon problem for exactly rung-dimerized spin ladder is brought up separately at all total spin sectors. At first a special duality transformation of the Schrödinger equation is found within general outlook. Then the problem is treated within Coordinate Bethe Ansatze. A straightforward approach is developed to obtain pure scattering states. At values $S=0$ and $S=3$ of total spin the Schrödinger equation has the form inherent in the XXZ chain. For $S=1,2$ solvability holds only in five previously found *completely integrable* cases. Nevertheless a partial $S=1$ Bethe solution always exists even for general non integrable model. Pure scattering states for all total spin sectors are presented explicitly.

1 Introduction

Among other gapped 1D systems spin ladders were intensively studied during the last 15 years experimentally, numerically and theoretically (see Refs. in [1]-[3]). The interest is accounted by their possibly relation to high temperature superconductivity, variety of static and dynamical properties and even an existence of several reliable compounds.

In the pioneering paper [4] a spin ladder was suggested as a double spin chain with Heisenberg interactions both across and along the chains direction namely rung and leg exchanges related to the couplings J_r and J_l. It was also pointed that the case,

$$J_r \gg J_l, \tag{1}$$

has a principle interest because it belongs to the so called *rung-dimerized phase* in which almost all spins are coupled into rung-singlets (rung-dimers). In the purely Heisenberg reference model this phase becomes exact only for $J_l = 0$. However it is always assumed that under the condition (1) the physical picture does not change in common.

Soon it became clear that spin ladder Hamiltonian also admit a term related to diagonal Heisenberg coupling as well as four spin terms [5]. At a first sight these new interactions seemed to be complications for a theoretical analysis. However even in [6] it was noted that a special linear condition (the Eq. (22) of the present paper) on the former and new coupling constants guarantees (for rather big J_r) *exactness* of the rung-dimerized ground state. Besides in this case all one- and two-magnon states also may be obtained in explicit form [6],[7].

Unfortunately the rung-dimerization condition (22) has no reliable atomic level interpretation, so there is no physical reason to postulate it. Nevertheless it seems reasonable to suppose that for strong rung exchange any deviations from the exact rung-dimerized picture should be small and may be evaluated perturbatively. (In more detail this question will be studied in a forthcoming paper.) Under this point of view exactly rung-dimerized spin ladders are the best reference models for treating the whole rung-dimerized phase.

Some static and dynamic zero-temperature properties of exactly rung-dimerized spin ladders were studied in a series of papers [7]-[10]. Due to the gap it succeeded to describe Raman scattering [7], magnetic phase transition [8] and (for asymmetric ladders) magnon decay [9],[10] utilizing only one- and two-magnon spectrums. Three-magnon problem is less actual for the $T = 0$ physics (see however the papers [11],[12] devoted to the $S = 1$ Haldane chain and $O(3)$ nonlinear σ-model).

Advancement into the $T > 0$ region needs a knowledge of the whole spectrum [3],[16]. However such level of clearness may be achieved only for a rather limited list of the so called integrable models [3],[13]-[17]. The latter besides are significant in heat transport phenomena [18].

But how to find an integrable model? How it may be identified from a overwhelming majority of nonintegrable ones? The most direct way is to express a treating Hamiltonian density as a derivative of the corresponding R-matrix which satisfy the Yang-Baxter equa-

tion. Solvability of this problem is governed by the Reshetikhin condition [17],[19],[20]. If the latter is satisfied for a given local Hamiltonian density then the corresponding R-matrix rather exist and may be obtained by an analysis of power series [20],[21],[22] or by some Yang-Baxterization ansatze [23]. In the present paper we suggest an alternative approach based on solvability of the three magnon problem in a framework of the Coordinate Bethe Ansatze (CBA) [24].

The essence of the CBA method [13],[14] is an assumption that any many-particle wave functions is in fact a linear combination of terms produced by multiplications of one-particle exponents. Namely, for a rung-dimerized spin ladder the one-magnon wave function $\psi(n) = \mathrm{e}^{ikn}$ [6] is parameterized by a real $0 \le k < 2\pi$ (the wave number) and depends on an integer n (position of the triplet rung). A two magnon wave function $\psi(m,n)$ $(m < n)$ is linear combination of two exponents $\mathrm{e}^{i(k_1 m + k_2 n)}$ and $\mathrm{e}^{i(k_2 m + k_1 n)}$ [7] and so depends on a pair of non equal parameters k_1 and k_2. For a scattering state they both are real and one may put

$$0 \le k_1 < k_2 < 2\pi, \tag{2}$$

while for a bound state they are complex conjugate

$$k_2 = \bar{k}_1. \tag{3}$$

In this light it is seems reasonable to search for representation of multi-magnon wave functions as sums of the Bethe exponents. However even a subsequent development of this approach to the three-magnon sector dashes on the problem of non integrability.

In order to reveal an origin of this obstacle let us at first turn back to a two-magnon state. Total quasimomentum (wave number) and energy of the latter are the sums

$$k = k_1 + k_2, \quad E(k_1, k_2) = E_{magn}(k_1) + E_{magn}(k_2), \tag{4}$$

where $E_{magn}(k)$ a single magnon energy. It is significant that under the conditions (2) or (3) the mapping

$$k_1, k_2 \longrightarrow k, E, \tag{5}$$

given by (4) is uniqually (up to an exchange $k_1 \leftrightarrow k_2$) reversible. However for three magnons the situation is drastically different. Indeed a system of relations

$$k = k_1 + k_2 + k_3, \quad E = E_{magn}(k_1) + E_{magn}(k_2) + E_{magn}(k_3), \tag{6}$$

can define an infinite number of triples (k_1, k_2, k_3). As a result a three-magnon wave function related to the pair (k, E) should contain in general an *infinite* number of exponential terms related to different solutions of the system (6). Evidently such three magnon problem is practically unsolvable.

The above obstacle may be overcame by existence a first integral (a translationary invariant operator commuting with the Hamiltonian) which produces the third condition additional to (6). An integrable system has an infinite number of such commuting in pairs first integrals and may be solved in all multi-particle sectors. It is significant that within the CBA a difference between integrability and non-integrability manifests just at the three particle level. As a consequence of this fact one may consider solvability the of three-particle problem as an alternative integrability test.

In the present paper we study three magnon sector of a rung-dimerized symmetric spin ladder. At first we briefly analyze the problem in general outlook and only afterwards turn to CBA. Motivation of such approach is the following argumentation. Usually CBA is treated as a successful ad hoc conjecture which allows to obtain in a rather straightforward manner all multi-particle states for a given an quantum integrable model. However the reference one is not integrable at general values of coupling constants. As a result (it will be shown below in detail) the CBA approach is applicable only in five special integrable cases.

The calculations are performed separately in the sectors $S = 0, 1, 2$ (the $S = 3$ sector is similar to the $S = 0$ one) of total spin. At $S = 0$ ($S = 3$) the system of equations on Bethe amplitudes has a well known form inherent in the XXZ spin chain and so is completely solvable for all values of coupling constants. For $S = 1$ and $S = 2$ a complete solvability takes place only in the five integrable cases obtained earlier [21] within the Yang-Baxter framework. However even in the general nonintegrable case there is a special (very complicated) solution in the $S = 1$ sector. Its interpretation remains unclear.

The plan of the paper is the following. In Sect. 2 we represent the spin ladder Hamiltonian in the most tractable form for which the rung-dimerized condition is evident. In Sect. 3 we show that the Bethe form of the two-magnon wave function readily follows from a straightforward treatment of the Shrödinger equation. In Sect.4 treating within general framework the $S = 0$ ($S = 3$) sector we reveal a duality transformation of wave function (generalized in Sect. 5 and 6 for $S = 1, 2$) and show that the Bethe Ansatze readily follows from the factorized (Fourier) substitution. We also obtain a classification (generalized in Sect. 5 and 6 for $S = 1, 2$) of Bethe three-magnon states related to complex wave numbers. Pure scattering states obtained within a straightforward approach developed in Sect. 4,5,6 are presented in the Appendix. In Sect. 7 we show that the revealed CBA solvability is in one to one correspondence with the integrability revealed earlier within the Yang-Baxter framework [21]. We also present the corresponding R-matrices. In Sect. 8 within CBA we describe action of the S_3 permutation group in all

total spin sectors. This symmetry as well as duality described in Sect. 5 and 6 is used in the Appendix for more compact representation of Bethe states.

Since the ground state of the model has a simple factorized form we treat it only in the infinite volume limit. Analogous approach to the ferromagnetic XXZ chain was developed in [14].

2 The spin ladder Hamiltonian

Before presenting the spin ladder Hamiltonian let us introduce the following local operators

$$\begin{aligned}\mathbf{\Psi}_n &= \frac{1}{2}(\mathbf{S}_{1,n} - \mathbf{S}_{2,n}) - i[\mathbf{S}_{1,n} \times \mathbf{S}_{2,n}],\\ \tilde{\mathbf{\Psi}}_n &= \frac{1}{2}(\mathbf{S}_{1,n} - \mathbf{S}_{2,n}) + i[\mathbf{S}_{1,n} \times \mathbf{S}_{2,n}],\end{aligned} \tag{7}$$

(we use the notation $\tilde{\mathbf{\Psi}}_n$ instead of more convenient $\mathbf{\Psi}^*_n$ or $\mathbf{\Psi}^\dagger_n$ only in order to avoid such rather cumbersome notations as $(\mathbf{\Psi}^a_n)^*$). Here $\mathbf{S}_{1,n}$ and $\mathbf{S}_{2,n}$ are local spin operators associated with n-th rung. They may be expressed from $\mathbf{\Psi}_n$ and $\tilde{\mathbf{\Psi}}_n$ as follows

$$\begin{aligned}\mathbf{S}_{1,n} &= \frac{1}{2}\Big(\mathbf{\Psi}_n + \tilde{\mathbf{\Psi}}_n - i[\tilde{\mathbf{\Psi}}_n \times \mathbf{\Psi}_\mathbf{n}]\Big),\\ \mathbf{S}_{2,n} &= \frac{1}{2}\Big(-\mathbf{\Psi}_n - \tilde{\mathbf{\Psi}}_n - i[\tilde{\mathbf{\Psi}}_n \times \mathbf{\Psi}_\mathbf{n}]\Big).\end{aligned} \tag{8}$$

The representation (8) is similar to the one suggested in [25] but in fact is not identical to it. Really the analogs of $\mathbf{\Psi}_n$ and $\tilde{\mathbf{\Psi}}_n$ treated in [25] act in an extended vector space. That is why for example the "inverse" representation (7) fails for them.

It may be readily proved that

$$[\mathbf{\Psi}_n, Q_n] = \mathbf{\Psi}_n, \quad [\tilde{\mathbf{\Psi}}_n, Q_n] = -\tilde{\mathbf{\Psi}}_n, \tag{9}$$

where

$$Q_n = \frac{1}{2}\mathbf{S}^2_n, \quad \mathbf{S}_n = \mathbf{S}_{1,n} + \mathbf{S}_{2,n}, \tag{10}$$

Let $|0\rangle_n$ and $|1\rangle_n$ be correspondingly singlet and triplet states associated with n-th rung. From (10) follows that

$$Q_n|0\rangle_n = 0, \qquad Q_n|1\rangle_n = |1\rangle_n, \tag{11}$$

so the local operator Q_n is projector on the n-th rung triplet sector. Then according to Eq. (9) the two triples $\tilde{\mathbf{\Psi}}_n$ and $\mathbf{\Psi}_n$ may be treated as rung-triplet creation-annihilation operators. Namely the tripe $|1\rangle_n^a$ $(a = x, y, z)$ for which

$$\tilde{\mathbf{\Psi}}_n^a|0\rangle_n = |1\rangle_n^a, \quad \tilde{\mathbf{\Psi}}_n^a|1\rangle_n^b = 0, \quad \mathbf{\Psi}_n^a|0\rangle_n = 0, \quad \mathbf{\Psi}_n^a|1\rangle_n^b = \delta_{ab}|0\rangle_n, \tag{12}$$

gives the following representation of the total rung-spin

$$\mathbf{S}_n^a|1\rangle_n^b = i\epsilon_{abc}|1\rangle_n^c, \tag{13}$$

(ϵ_{abc} is the Levi-Chivita tensor). Parallel with (12) we shall use the triple

$$|1\rangle_n^j = \tilde{\mathbf{\Psi}}_n^j|0\rangle_n, \quad \mathbf{S}_n^z|1\rangle_n^j = j|1\rangle_n^j, \quad j = -1, 0, 1. \tag{14}$$

related to operators

$$\tilde{\mathbf{\Psi}}_n^{\pm 1} \equiv \frac{1}{\sqrt{2}}\Big(\tilde{\mathbf{\Psi}}_n^x \pm i\tilde{\mathbf{\Psi}}_n^y\Big), \quad \tilde{\mathbf{\Psi}}_n^0 \equiv \tilde{\mathbf{\Psi}}_n^z. \tag{15}$$

It seems reasonable to represent the Hamiltonian density $H_{n,n+1}$ for general spin ladder Hamiltonian

$$\hat{H} = \sum_n H_{n,n+1}, \tag{16}$$

in the following form

$$\begin{aligned} H_{n,n+1} &= J_1(Q_n + Q_{n+1}) + J_2(\mathbf{\Psi}_n \cdot \tilde{\mathbf{\Psi}}_{n+1} + \tilde{\mathbf{\Psi}}_n \cdot \mathbf{\Psi}_{n+1}) \\ &+ J_3 Q_n Q_{n+1} + J_4 \mathbf{S}_n \cdot \mathbf{S}_{n+1} + J_5(\mathbf{S}_n \cdot \mathbf{S}_{n+1})^2 \\ &+ J_6(\tilde{\mathbf{\Psi}}_n \cdot \tilde{\mathbf{\Psi}}_{n+1} + \mathbf{\Psi}_n \cdot \mathbf{\Psi}_{n+1}). \end{aligned} \tag{17}$$

Up to a constant this representation is equivalent to the standard one [1]-[6]

$$H_{n,n+1} = J_r H^r_{n,n+1} + J_l H^l_{n,n+1} + J_d H^d_{n,n+1} + J_{rr} H^{rr}_{n,n+1} + J_{ll} H^{ll}_{n,n+1} + J_{dd} H^{dd}_{n,n+1}, \tag{18}$$

where

$$\begin{aligned} H^r_{n,n+1} &= \frac{1}{2}(\mathbf{S}_{1,n} \cdot \mathbf{S}_{2,n} + \mathbf{S}_{1,n+1} \cdot \mathbf{S}_{2,n+1}), \quad H^l_{n,n+1} = \mathbf{S}_{1,n} \cdot \mathbf{S}_{1,n+1} + \mathbf{S}_{2,n} \cdot \mathbf{S}_{2,n+1}, \\ H^d_{n,n+1} &= \mathbf{S}_{1,n} \cdot \mathbf{S}_{2,n+1} + \mathbf{S}_{2,n} \cdot \mathbf{S}_{1,n+1}, \quad H^{rr}_{n,n+1} = (\mathbf{S}_{1,n} \cdot \mathbf{S}_{2,n})(\mathbf{S}_{1,n+1} \cdot \mathbf{S}_{2,n+1}), \\ H^{ll}_{n,n+1} &= (\mathbf{S}_{1,n} \cdot \mathbf{S}_{1,n+1})(\mathbf{S}_{2,n} \cdot \mathbf{S}_{2,n+1}), \quad H^{dd}_{n,n+1} = (\mathbf{S}_{1,n} \cdot \mathbf{S}_{2,n+1})(\mathbf{S}_{2,n} \cdot \mathbf{S}_{1,n+1}), \end{aligned} \tag{19}$$

and

$$J_1 = \frac{1}{4}\Big(2J_r - 3J_{rr} - J_{ll} - J_{dd}\Big),$$

$$
\begin{aligned}
J_2 &= \frac{1}{8}\Big(4(J_l - J_d) + J_{ll} - J_{dd}\Big),\\
J_3 &= J_{rr},\\
J_4 &= \frac{1}{8}\Big(4(J_l + J_d) + J_{ll} + J_{dd}\Big),\\
J_5 &= \frac{1}{4}\Big(J_{ll} + J_{dd}\Big),\\
J_6 &= \frac{1}{8}\Big(4(J_l - J_d) - J_{ll} + J_{dd}\Big).
\end{aligned} \tag{20}
$$

It was suggested in [5] that only the case

$$J_{rr} = J_{ll} = -J_{dd}, \tag{21}$$

(or equivalently $J_5 = 0$, $J_6 = J_2 - J_3/2$) has a reliable interest. However since spin ladders with failed condition (21) also are currently studied [3] and we shall not require it.

From (9) and (17) directly follows that for

$$J_6 = 0 \Leftrightarrow J_{ll} - J_{dd} = 4(J_l - J_d), \tag{22}$$

(triplet-rungs pair creation-annihilation processes are suppressed) there holds

$$[\hat{H}, \hat{Q}] = 0. \tag{23}$$

Here the global operator

$$\hat{Q} = \sum_n Q_n, \tag{24}$$

according to (11) may be treated as a number operator for rung-triplets. The commutation relation (23) results in splitting of the Hilbert space on an infinite sum of eigenspaces related to different eigenvalues of $\hat{Q}$. In particularly for rather strong J_1 the (zero energy) ground state of the model has a simple tensor-product form [6]

$$|0\rangle = \prod_n \otimes |0\rangle_n. \tag{25}$$

At the same time the physical Hilbert space is subdivided into a direct sum of magnon sectors

$$\mathcal{H} = \sum_{m=0}^{\infty} \mathcal{H}^m, \quad \hat{Q}|_{\mathcal{H}^m} = m. \tag{26}$$

Only this special case (Eq. (22) and rather strong J_1) will be studied in the present paper. Additionally we shall imply that $J_2 \neq 0$. The completely diagonal frustrated model related to $J_2 = 0$ or equivalently $J_d = J_l$ (in this case the Hamiltonian density (17)

may be expressed only in terms of Q_n and $\mathbf{S}_n$) was studied in details in [26]. Besides one may assume that

$$J_2 > 0 \Leftrightarrow J_l > J_d. \tag{27}$$

Indeed the case $J_2 < 0$ can be reduced to (27) by use of the following exchange of the coupling constants

$$J_l \leftrightarrow J_d, \qquad J_{ll} \leftrightarrow J_{dd}, \tag{28}$$

related to permutation of spins on all even (odd) rungs.

3 One- and two-magnon states

Taking into account (17), (11), (12) and (14) one gets the local formulas

$$\begin{aligned} H_{n,n+1} \ldots |1\rangle_n |0\rangle_{n+1} \ldots &= J_1 \ldots |1\rangle_n |0\rangle_{n+1} \ldots + J_2 \ldots |0\rangle_n |1\rangle_{n+1} \ldots, \\ H_{n-1,n} \ldots |0\rangle_{n-1} |1\rangle_n \ldots &= J_1 \ldots |0\rangle_{n-1} |1\rangle_n \ldots + J_2 \ldots |1\rangle_{n-1} |0\rangle_n \ldots, \end{aligned} \tag{29}$$

and

$$\begin{aligned} H_{n,n+1} \ldots |1\rangle_n^a |1\rangle_{n+1}^a \ldots &= \varepsilon_0 \ldots |1\rangle_n^a |1\rangle_{n+1}^a \ldots, \\ H_{n,n+1} \epsilon_{abc} \ldots |1\rangle_n^b |1\rangle_{n+1}^c \ldots &= \varepsilon_1 \epsilon_{abc} \ldots |1\rangle_n^b |1\rangle_{n+1}^c \ldots, \quad (a,b,c = x,y,z) \\ H_{n,n+1} \ldots |1\rangle_n^+ |1\rangle_{n+1}^+ \ldots &= \varepsilon_2 \ldots |1\rangle_n^+ |1\rangle_{n+1}^+ \ldots \end{aligned} \tag{30}$$

Here

$$\varepsilon_S \equiv 2(J_1 + J_2 \Delta_S), \tag{31}$$

and

$$\begin{aligned} \Delta_0 &= \frac{J_3 - 2J_4 + 4J_5}{2J_2} = \frac{4(J_d - 2J_l) + 2J_{rr} + 3J_{ll}}{4(J_l - J_d)}, \\ \Delta_1 &= \frac{J_3 - J_4 + J_5}{2J_2} = \frac{4(J_{rr} - J_l) + J_{ll}}{8(J_l - J_d)}, \\ \Delta_2 &= \frac{J_3 + J_4 + J_5}{2J_2} = \frac{4(2J_d - J_l) + 4J_{rr} + 3J_{ll}}{8(J_l - J_d)}. \end{aligned} \tag{32}$$

From (30) may be readily obtained a useful formula

$$\begin{aligned} H_{n,n+1} \ldots |1\rangle_n^a |1\rangle_{n+1}^b \ldots &= \Big(2J_1 + J_3 + J_5\Big) \ldots |1\rangle_n^a |1\rangle_{n+1}^b \ldots + J_4 \ldots |1\rangle_n^b |1\rangle_{n+1}^a \ldots \\ &+ \delta_{ab} \Big(J_5 - J_4\Big) \ldots |1\rangle_n^c |1\rangle_{n+1}^c \ldots, \quad (a,b,c = x,y,z). \end{aligned} \tag{33}$$

Turning to excitation states we notice that an explicit form of a one-magnon state

$$|1,k\rangle = \sum_n e^{ikn}\Big(\prod_{m=-\infty}^{n-1} \otimes|0\rangle_m\Big) \otimes |1\rangle_n \otimes \Big(\prod_{m=n+1}^{\infty} \otimes|0\rangle_m\Big). \tag{34}$$

directly follows from (23) and translation symmetry

$$\hat{P}|1,k\rangle = \mathrm{e}^{-ik}|1,k\rangle. \tag{35}$$

Here $\hat{P}$ is the translation operator

$$\hat{P}\prod \otimes|\chi(n)\rangle_n = \prod \otimes|\chi(n)\rangle_{n+1}, \qquad \chi(n) = 0,1. \tag{36}$$

The corresponding dispersion

$$E_{magn}(k) = 2(J_1 + J_2\cos k), \tag{37}$$

readily follows from (29).

Since the $\hat{Q} = 2$ sector is subdivided on the total spin $S = 0,1,2$ subsectors we denote at once a two-magnon state with total spin S and wave vector k as $|2,S,k\rangle$. The following general representations for the two-magnon states

$$\begin{aligned}
|2,0,k\rangle &= \sum_{m<n} \mathrm{e}^{ik(m+n)/2} a_0(k,n-m)\ldots|1\rangle_m^a\ldots|1\rangle_n^a\ldots,\\
|2,1,k\rangle^a &= \varepsilon_{abc}\sum_{m<n} \mathrm{e}^{ik(m+n)/2} a_1(k,n-m)\ldots|1\rangle_m^b\ldots|1\rangle_n^c\ldots,\\
|2,2,k\rangle^{+2} &= \sum_{m<n} \mathrm{e}^{ik(m+n)/2} a_2(k,n-m)\ldots|1\rangle_m^+\ldots|1\rangle_n^+\ldots,
\end{aligned} \tag{38}$$

agree with the rotational and translation (35) symmetries. From Eq. (38) by "..." we denote an appropriate tensor product of *rung-singlets* (similar to products in (34)). For simplicity the $S = 2$ sector is represented in (38) by the $\mathbf{S}^z = +2$ states. Besides we suggest that the reduced wave function $a_S(k,n)$ should be bounded

$$\sup_n a_S(k,n) < \infty. \tag{39}$$

The Schrödinger equation for $a_S(k,n)$ has different forms at $n > 1$ and $n = 1$. In the former case Eqs. (29) and (30) give

$$4J_1 a_S(k,n) + 2J_2\cos\frac{k}{2}[a_S(k,n-1) + a_S(k,n+1)] = Ea_S(k,n), \tag{40}$$

while in the latter

$$(2J_1 + \varepsilon_S)a_S(k,1) + 2J_2 \cos\frac{k}{2} a_S(k,2) = E(k)a_S(k,1). \tag{41}$$

It is convenient to rewrite Eq. (41) in the form of Eq. (40) [13] by continuing $a_S(k,n)$ into unphysical region $n = 0$. Comparing (40) and (41) one conclude that this trick entails a Bethe condition

$$\Delta_S a_S(k,1) = \cos\frac{k}{2} a_S(k,0). \tag{42}$$

The system (40) (considered now for $n \geq 1$) and (42) together with the restriction (39) allows us to obtain entire $a_S(k,n)$ in a straightforward manner. Indeed representing Eq. (40) in an equivalent matrix form

$$\begin{pmatrix} a_S(k,n+1) \\ a_S(k,n) \end{pmatrix} = \mathcal{F}(\kappa) \begin{pmatrix} a_S(k,n) \\ a_S(k,n-1) \end{pmatrix}, \tag{43}$$

where

$$\mathcal{F}(\kappa) = \begin{pmatrix} 2\kappa & -1 \\ 1 & 0 \end{pmatrix}, \quad \kappa = \frac{E - 4J_1}{4J_2 \cos k/2}, \tag{44}$$

and taking $a_S(k,1) : a_S(k,0)$ from (42) one consequently obtains (up to a constant factor) using (43) the rest of $a_S(k,n)$ at $n = 2, 3, \ldots$ In following we shall study this problem in detail considering separately three regions $|\kappa| < 1$, $|\kappa| > 1$ and $|\kappa| = 1$.

For $|\kappa| \neq 1$ the matrix $\mathcal{F}(\kappa)$ has two different eigenvalues

$$\Lambda_\pm(\kappa) = \kappa \pm \sqrt{\kappa^2 - 1}, \tag{45}$$

related to eigenvectors

$$\xi_\pm(\kappa) = \begin{pmatrix} \Lambda_\pm(\kappa) \\ 1 \end{pmatrix}. \tag{46}$$

At $|\kappa| < 1$ it is more convenient to use the following representation

$$\Lambda_\pm(\kappa) = \mathrm{e}^{\pm iq}, \quad \kappa = \cos q, \quad 0 < q < \pi. \tag{47}$$

According to (43) a decomposition

$$\begin{pmatrix} a_S(k,1) \\ a_S(k,0) \end{pmatrix} = c_+ \xi_+(\kappa) + c_- \xi_-(\kappa), \tag{48}$$

($c_\pm$ are some coefficients) results in

$$\begin{pmatrix} a_S(k,n+1) \\ a_S(k,n) \end{pmatrix} = \Lambda_+^n(\kappa)c_+\xi_+(\kappa) + \Lambda_-^n(\kappa)c_-\xi_-(\kappa). \tag{49}$$

or equivalently

$$a_S^{scatt}(k,q,n) = \cos\frac{k}{2}\sin qn - \Delta_S \sin q(n-1). \tag{50}$$

The expression (50) obviously agrees with (39) and (as it readily follows from (29) and (30)) corresponds to dispersion

$$E_{scatt}(k,q) = 4\Big(J_1 + J_2\cos q\cos\frac{k}{2}\Big). \tag{51}$$

According to the following formulas

$$\begin{aligned} e^{ik(m+n)/2}a_S^{scatt}(k,q,n-m) &= \frac{1}{2i}\Big[C_{S,12}e^{i(k_1 m + k_2 n)} - C_{S,21}e^{i(k_2 m + k_1 n)}\Big], \\ E_{scatt}(k,q) &= E_{magn}(k_1) + E_{magn}(k_2), \end{aligned} \tag{52}$$

where

$$\frac{k}{2} - q = k_1 < k_2 = \frac{k}{2} + q, \quad q = \frac{k_2 - k_1}{2}, \tag{53}$$

and

$$C_{S,ab} = \cos\frac{k_a + k_b}{2} - \Delta_S e^{i(k_a - k_b)/2}, \tag{54}$$

one may associate (50) with a scattering wave function of two magnons with wave vectors k_1 and k_2 reduced to the center mass frame.

For $|\kappa| > 1$ both the eigenvalues (45) are real. More specifically at $\pm\kappa > 1$ there should be $|\Lambda_\mp(\kappa)| < 1 < |\Lambda_\pm(\kappa)|$ and the representation (49) agree with (39) only for $c_\pm = 0$. According to (42), (46) and (48) in both the cases the eigenvalue remaining in (49) is $(\cos k/2)/\Delta_S$. So one gets

$$\begin{aligned} a_S^{bound}(k,n) &= \Big(\frac{\cos k/2}{\Delta_S}\Big)^n, \\ E_{bound}(S,k) &= 2\Big(2J_1 + J_2\Delta_S + \frac{J_2}{\Delta_S}\cos^2\frac{k}{2}\Big). \end{aligned} \tag{55}$$

This solution exists only for

$$-|\Delta_S| < \cos\frac{k}{2} < |\Delta_S|, \tag{56}$$

and according to the formulas

$$\mathrm{e}^{ik(m+n)/2}a_S^{bound}(k,n-m)=\mathrm{e}^{i(k_1m+k_2n)},$$
$$E_{bound}(S,k)=E_{magn}(k_1)+E_{magn}(k_2), \tag{57}$$

where

$$k_1=\frac{k}{2}-iv,\quad k_2=\frac{k}{2}+iv,\quad v=\ln\Big(\frac{\Delta_S}{\cos k/2}\Big), \tag{58}$$

may be associated with a two-magnon bound state wave function reduced to the center mass frame. Both (52) and (57) reproduce the Bethe Ansatze calculation presented in [7].

For $\kappa^2=1$ the matrix $\mathcal{F}(\kappa)$ has an eigenvector ξ_0 and an adjoint vector $\tilde{\xi}_0$

$$\xi_0=\left(\begin{array}{c}\kappa\\ 1\end{array}\right),\quad \tilde{\xi}_0=\left(\begin{array}{c}1\\ 0\end{array}\right), \tag{59}$$

for which

$$\mathcal{F}(\kappa)\xi_0(\kappa)=\kappa\xi_0(\kappa),\quad \mathcal{F}(\kappa)\tilde{\xi}_0(\kappa)=\kappa\tilde{\xi}_0(\kappa)+\xi_0(\kappa). \tag{60}$$

Taking into account (60) and (42) one gets the following decomposition

$$\left(\begin{array}{c}a_S(k,1)\\ a_S(k,0)\end{array}\right)=\Delta_S\xi_0(\kappa)+(\cos\frac{k}{2}-\kappa\Delta_S)\tilde{\xi}_0(\kappa). \tag{61}$$

The resulting wave function

$$a_S(k,\kappa,n)=n\kappa^{n-1}(\cos\frac{k}{2}-\kappa\Delta_S)+\kappa^n\Delta_S. \tag{62}$$

agrees with (39) only on the appropriate bound of the interval (56), namely for

$$\cos\frac{k}{2}=\kappa\Delta_S. \tag{63}$$

The solution (62) may be obtained from both (50) or (55) in the limit $|\kappa|\to 1$. Indeed despite the wave function (50) turns to zero at $q=0,\pi$ the ratio $a^{scatt}(k,q,n)/\sin q$ remains finite and gives (62) as a limit value. Analogously using the formula $(1+\epsilon)^n=1+n\epsilon+o(\epsilon)$ one can obtain (62) from (55).

4 S = 0 and S = 3 three-magnon sectors

Representing at once a $S=0$ state in general translationary covariant form

$$\begin{aligned}|3,0,k\rangle &= \epsilon_{abc}\sum_{m<n<p}\mathrm{e}^{ik(m+n+p)/3}b_0(k,n-m,p-n)\\ &\qquad \ldots|1\rangle_m^a\ldots|1\rangle_n^b\ldots|1\rangle_p^c\ldots,\end{aligned} \tag{64}$$

one readily obtains from (29) and (33) the Schrödinger equation for the reduced wave function $b_0(k,m,n)$ both for $m,n>1$

$$\begin{gathered}6J_1 b_0(k,m,n) + J_2[\mathrm{e}^{-ik/3} b_0(k,m+1,n) + \mathrm{e}^{ik/3} b_0(k,m-1,n)\\ +\mathrm{e}^{-ik/3} b_0(k,m-1,n+1) + \mathrm{e}^{ik/3} b_0(k,m+1,n-1) + \mathrm{e}^{-ik/3} b_0(k,m,n-1)\\ +\mathrm{e}^{ik/3} b_0(k,m,n+1)] = E b_0(k,m,n),\end{gathered} \tag{65}$$

and $m,n=1$

$$\begin{gathered}(4J_1+\varepsilon_1) b_0(k,1,n) + J_2[\mathrm{e}^{-ik/3} b_0(k,1,n-1) + \mathrm{e}^{ik/3} b_0(k,1,n+1)\\ +\mathrm{e}^{ik/3} b_0(k,2,n-1) + \mathrm{e}^{-ik/3} b_0(k,2,n)] = E b_0(k,1,n),\\ (4J_1+\varepsilon_1) b_0(k,m,1) + J_2[\mathrm{e}^{ik/3} b_0(k,m-1,1) + \mathrm{e}^{-ik/3} b_0(k,m+1,1)\\ +\mathrm{e}^{-ik/3} b_0(k,m-1,2) + \mathrm{e}^{ik/3} b_0(k,m,2)] = E b_0(k,m,1).\end{gathered} \tag{66}$$

Reduction of (66) into (65) results in a system of Bethe conditions,

$$\begin{aligned}2\Delta_1 b_0(k,1,n) &= \mathrm{e}^{ik/3} b_0(k,0,n) + \mathrm{e}^{-ik/3} b_0(k,0,n+1),\\ 2\Delta_1 b_0(k,m,1) &= \mathrm{e}^{-ik/3} b_0(k,m,0) + \mathrm{e}^{ik/3} b_0(k,m+1,0).\end{aligned} \tag{67}$$

The pair (65) (considered for $m,n>0$) and (67) represents the Schrödinger equation for $b_0(k,m,n)$. It is invariant under the following duality transformation

$$\mathcal{D}(b_0(k,m,n)) = \bar{b}_0(k,n,m). \tag{68}$$

Autodual and anti-autodual solutions are related by multiplication on i.

As in the two-magnon case we suggest that the reduced wave function should be bounded

$$\sup_{m,n} b_0(k,m,n) < \infty. \tag{69}$$

Despite the system (65), (67) is linear a proper generalization of the straightforward matrix approach used in the previous section is unclear for it. Instead one may treat (65) by the Fourier substitution

$$\tilde{b}_0(k,m,n) = \varphi(k,m)\theta(k,n), \tag{70}$$

which results in the following two-parametric exponential solution

$$\tilde{b}_0(k,m,n) = \mathrm{e}^{i(\tilde{q}_1 m + \tilde{q}_2 n)}, \tag{71}$$

related to dispersion

$$E(k,\tilde{q}_1,\tilde{q}_2) = 2[3J_1 + J_2(\cos k_1 + \cos k_2 + \cos k_3)], \tag{72}$$

where

$$k_1 = \frac{k}{3} - \tilde{q}_1, \quad k_2 = \frac{k}{3} + \tilde{q}_1 - \tilde{q}_2, \quad k_3 = \frac{k}{3} + \tilde{q}_2. \tag{73}$$

Since

$$\mathrm{e}^{ik/3(m+n+p)}\tilde{b}_0(k, n-m, p-n) = \mathrm{e}^{i(k_1 m + k_2 n + k_3 p)}, \tag{74}$$

one can naturally associate (71) with the wave function of a triple of magnons related to wave numbers k_1, k_2 and k_3.

Instead of $\tilde{q}_{1,2}$ we shall mainly use the parameters

$$q_1 = \frac{k_2 - k_1}{2} = \tilde{q}_1 - \frac{\tilde{q}_2}{2}, \quad q_2 = \frac{k_3 - k_2}{2} = \tilde{q}_2 - \frac{\tilde{q}_1}{2}, \tag{75}$$

considering them as generalizations of the parameter q from Eq. (53). The pair $q_{1,2}$ is more convenient for representation of *pure scattering* states related to real $k_{1,2,3}$ with $0 \le k_1 < k_2 < k_3 \le 2\pi$ (a generalization of Eq. (2)) because the latter system of inequalities in terms of $q_{1,2}$ has a very simple form. Namely $0 < q_{1,2} < \pi$ and $0 < q_1 + q_2 < \pi$. Nevertheless due to a rather compact representation (71) the parameters $\tilde{q}_{1,2}$ still will be remained in exponential factors. They also will be used for classification of states with complex wave numbers (Eq. (82)).

When $\tilde{q}_{1,2}$ are complex numbers one have to treat carefully the condition (69) and take into account that the energy (72) must be real. These conditions result in

$$\Im(\tilde{q}_j) \ge 0, \quad j = 1, 2, \tag{76}$$

$$\begin{aligned}&\Big(\sin[k/3 - \Re(\tilde{q}_1)] - \sin[k/3 + \Re(\tilde{q}_1 - \tilde{q}_2)]\cosh\Im(\tilde{q}_2)\Big)\sinh\Im(\tilde{q}_1)\\ &= \Big(\sin[k/3 + \Re(\tilde{q}_2)] - \sin[k/3 + \Re(\tilde{q}_1 - \tilde{q}_2)]\cosh\Im(\tilde{q}_1)\Big)\sinh\Im(\tilde{q}_2).\end{aligned} \tag{77}$$

The dispersion (72) is invariant under permutations of k_1, k_2 and k_3 or equivalently under the following transformations of $\tilde{\mathbf{q}} \equiv (\tilde{q}_1, \tilde{q}_2)$

$$\begin{aligned}\omega_1(\tilde{\mathbf{q}}) &= (\tilde{q}_1, \tilde{q}_1 - \tilde{q}_2), \quad \omega_2(\tilde{\mathbf{q}}) = (-\tilde{q}_2, \tilde{q}_1 - \tilde{q}_2), \quad \omega_3(\tilde{\mathbf{q}}) = (-\tilde{q}_2, -\tilde{q}_1)\\ \omega_4(\tilde{\mathbf{q}}) &= (\tilde{q}_2 - \tilde{q}_1, -\tilde{q}_1), \quad \omega_5(\tilde{\mathbf{q}}) = (\tilde{q}_2 - \tilde{q}_1, \tilde{q}_2).\end{aligned} \tag{78}$$

In fact these formulas give a representation of the three-elements permutation group S_3. It may be readily checked that all ω_j are generated by ω_1 and ω_5. Namely,

$$\omega_2 = \omega_5 \cdot \omega_1, \quad \omega_3 = \omega_5 \cdot \omega_1 \cdot \omega_5 = \omega_1 \cdot \omega_5 \cdot \omega_1, \quad \omega_4 = \omega_1 \cdot \omega_5. \tag{79}$$

The symmetry (78) allows to generalize the solution (71) and suggest the following ansatze ($\mathbf{q} \equiv (q_1, q_2)$),

$$\begin{aligned} b_0(k, \mathbf{q}, m, n) &= A_1(k, \mathbf{q}) e^{i(\tilde{q}_1 m + \tilde{q}_2 n)} - A_2(k, \mathbf{q}) e^{i(\tilde{q}_1 m + (\tilde{q}_1 - \tilde{q}_2) n)} \\ &+ A_3(k, \mathbf{q}) e^{i(-\tilde{q}_2 m + (\tilde{q}_1 - \tilde{q}_2) n)} - A_4(k, \mathbf{q}) e^{-i(\tilde{q}_2 m + \tilde{q}_1 n)} \\ &+ A_5(k, \mathbf{q}) e^{i((\tilde{q}_2 - \tilde{q}_1) m - \tilde{q}_1 n)} - A_6(k, \mathbf{q}) e^{i((\tilde{q}_2 - \tilde{q}_1) m + \tilde{q}_2 n)}. \end{aligned} \tag{80}$$

For $\Im(\tilde{q}_{1,2}) = 0$ the expression (80) agree with (69) while Eq. (77) is satisfied identically. However even when one of $\tilde{q}_{1,2}$ has an imaginary part some of the amplitudes in (80) must turn to zero in order to ensure an agreement with (69). Besides according to (77) real and imaginary parts of $\tilde{q}_{1,2}$ should be interdependent. More specifically let us divide the sector $\Im(\tilde{q}_{1,2}) \geq 0$ on five subsectors

$$\begin{aligned} \mathcal{V}_1 &= [\Im(\tilde{q}_1) > 0,\, \Im(\tilde{q}_2) = 0], \quad \mathcal{V}_2 = [\Im(\tilde{q}_1) = 0,\, \Im(\tilde{q}_2) > 0], \\ \mathcal{V}_3 &= [0 < \Im(\tilde{q}_1) = \Im(\tilde{q}_2)], \quad \mathcal{V}_4 = [0 < \Im(\tilde{q}_1) < \Im(\tilde{q}_2)], \\ \mathcal{V}_5 &= [0 < \Im(\tilde{q}_2) < \Im(\tilde{q}_1)]. \end{aligned} \tag{81}$$

Let $\mathcal{J}_i$ will be corresponding to $\mathcal{V}_i$ set of l-s for which there should be $A_l(k, \mathbf{q}) = 0$. At the same time $\mathcal{Q}_i$ will be the corresponding additional condition on $\tilde{q}_{1,2}$ following from (77). For each i we may gather a triple $\mathcal{W}_i = [\mathcal{V}_i; \mathcal{J}_i; \mathcal{G}_i]$. A straightforward analysis based on Eqs. (77) and (80) results in the following classification

$$\begin{aligned} \mathcal{W}_1 &= [\Im(\tilde{q}_1) > 0,\, \Im(\tilde{q}_2) = 0;\, \{4,5,6\};\, \Re(\tilde{q}_2) = 2\Re(\tilde{q}_1)], \\ \tilde{\mathcal{W}}_1 &= [\Im(\tilde{q}_1) > 0,\, \Im(\tilde{q}_2) = 0;\, \{4,5,6\};\, \Re(\tilde{q}_2) = 2k/3 + \pi], \\ \mathcal{W}_2 &= [\Im(\tilde{q}_1) = 0,\, \Im(\tilde{q}_2) > 0;\, \{2,3,4\};\, \Re(\tilde{q}_1) = 2\Re(\tilde{q}_2)], \\ \tilde{\mathcal{W}}_2 &= [\Im(\tilde{q}_1) = 0,\, \Im(\tilde{q}_2) > 0;\, \{2,3,4\};\, \Re(\tilde{q}_1) = -2k/3 + \pi], \\ \mathcal{W}_3 &= [0 < \Im(\tilde{q}_1) = \Im(\tilde{q}_2);\, \{3,4,5\};\, \Re(\tilde{q}_2) = -\Re(\tilde{q}_1)], \\ \tilde{\mathcal{W}}_3 &= [0 < \Im(\tilde{q}_1) = \Im(\tilde{q}_2);\, \{3,4,5\};\, \Re(\tilde{q}_2) = \Re(\tilde{q}_1) + \pi - 2k/3], \\ \mathcal{W}_4 &= [0 < \Im(\tilde{q}_1) < \Im(\tilde{q}_2);\, \{2,3,4,5\};\, \text{Eq. (77)}], \\ \mathcal{W}_5 &= [0 < \Im(\tilde{q}_2) < \Im(\tilde{q}_1);\, \{3,4,5,6\};\, \text{Eq. (77)}]. \end{aligned} \tag{82}$$

Each Bethe state with complex $\tilde{q}$-s corresponds without fail to one of the $\mathcal{W}$-s presented in (82).

The system (65) (at $m, n \geq 1$), (67) exactly coincides with the well known one inherent in the XXZ model [13],[14]. Nevertheless we shall give its solution within the ansatze

(80) in order to illustrate the straightforward approach used in the next section for $S = 1$ and $S = 2$.

Let us begin with pure scattering states for which the duality transformation (68) results in

$$\mathcal{D}(A_l(k, \mathbf{q})) = \bar{A}_{l-3}(k, \mathbf{q}), \tag{83}$$

where $A_l(k, \mathbf{q}) \equiv A_{l+6}(k, \mathbf{q})$ for $l = -2, -1, 0$. The $S = 1$ and $S = 2$ analogs of this formula are be used in Appendix for enumeration of three-magnon Bethe states.

Substitution of (80) into (67) produces a linear system on the amplitudes $A_l(k, \mathbf{q})$

$$\sum_{l=1}^{6} M_{il}^{(0)}(k, \mathbf{q}) A_l(k, \mathbf{q}) = 0, \tag{84}$$

where nonzero entries of the 6×6 matrix $M^{(0)}(k, \mathbf{q})$ are

$$\begin{aligned}
M_{11}^{(0)}(k, \mathbf{q}) &= -M_{45}^{(0)}(k, \mathbf{q}) = Z(k, \omega_5(\mathbf{q}), \Delta_1), \\
M_{22}^{(0)}(k, \mathbf{q}) &= -M_{56}^{(0)}(k, \mathbf{q}) = Z(k, \mathbf{q}, \Delta_1), \\
M_{33}^{(0)}(k, \mathbf{q}) &= -M_{61}^{(0)}(k, \mathbf{q}) = Z(k, \omega_1(\mathbf{q}), \Delta_1), \\
M_{44}^{(0)}(k, \mathbf{q}) &= -M_{12}^{(0)}(k, \mathbf{q}) = Z(k, \omega_2(\mathbf{q}), \Delta_1), \\
M_{55}^{(0)}(k, \mathbf{q}) &= -M_{23}^{(0)}(k, \mathbf{q}) = Z(k, \omega_3(\mathbf{q}), \Delta_1), \\
M_{66}^{(0)}(k, \mathbf{q}) &= -M_{34}^{(0)}(k, \mathbf{q}) = Z(k, \omega_4(\mathbf{q}), \Delta_1).
\end{aligned} \tag{85}$$

Here

$$Z(k, \mathbf{q}, \Delta) = \cos\left(\frac{k + q_2 - q_1}{3}\right) - \Delta \mathrm{e}^{i(q_1+q_2)}, \tag{86}$$

while according to (75) and (78)

$$\begin{aligned}
\omega_1(\mathbf{q}) &= (q_1 + q_2, -q_2), \quad \omega_2(\mathbf{q}) = (-q_1 - q_2, q_1), \quad \omega_3(\mathbf{q}) = (-q_2, -q_1), \\
\omega_4(\mathbf{q}) &= (q_2, -q_1 - q_2), \quad \omega_5(\mathbf{q}) = (-q_1, q_1 + q_2).
\end{aligned} \tag{87}$$

Since

$$\det M^{(0)}(k, \mathbf{q}) = \prod_{n=1..6} M_{nn}^{(0)}(k, \mathbf{q}) - \prod_{n=1..6} M_{n,n+1}^{(0)}(k, \mathbf{q}) = 0, \tag{88}$$

(here $M_{67}^{(0)}(k, \mathbf{q}) \equiv M_{61}^{(0)}(k, \mathbf{q})$) the matrix system (84) is solvable. Namely

$$A_l(k, \mathbf{q}) = \prod_{i=1}^{3} M_{l-i,l-i}^{(0)}(k, \mathbf{q}), \tag{89}$$

where $M_{ll}^{(0)}(k, \mathbf{q}) \equiv M_{l+6,l+6}^{(0)}(k, \mathbf{q})$ for $l = -2, -1, 0$.

States with complex $\tilde{q}_{1,2}$ may be obtained from (89) by analytic continuation with regard to conditions presented in the list (82). It may be readily shown by a straightforward calculations that there are no solutions related to $\tilde{\mathcal{W}}_{1,2,3}$ and $\mathcal{W}_{4,5}$. This statement is a special (related to the three-magnon sector) confirmation of the string hypothesis proved for the XXZ chain [13],[14].

It may be readily proved that the $S=3$ case is analogous to the $S=0$ one. It is only necessary to improve the representation (64) (in order to obtain the state with total spin $S=3$) and replace everywhere Δ_1 on Δ_2.

5 S = 1 three-magnon sector

General $S=1$ three-magnon state has the following representation

$$|3,1,k\rangle^a = \sum_{m<n<p} \mathrm{e}^{ik(m+n+p)/3}\Big[b_1^{(1)}(k,n-m,p-n)$$
$$\ldots|1\rangle^a_m\ldots|1\rangle^b_n\ldots|1\rangle^b_p\ldots + b_1^{(2)}(k,n-m,p-n)\ldots|1\rangle^b_m\ldots|1\rangle^a_n\ldots|1\rangle^b_p\ldots$$
$$+b_1^{(3)}(k,n-m,p-n)\ldots|1\rangle^b_m\ldots|1\rangle^b_n\ldots|1\rangle^a_p\ldots\Big], \tag{90}$$

and depends on the three wave functions $b_1^{(1,2,3)}(k,m,n)$. At $m,n>1$ the Schrödinger equation for $b_1^{(1,2,3)}(k,m,n)$ separates on three independent linear subsystems of the form (65) (one have only to replace $b_0(k,m,n)$ on $b_1^{(1,2,3)}(k,m,n)$). However for $m,n=1$ one gets

$$\Big(6J_1+J_2+\frac{3}{2}J_3\Big)b_1^{(1)}(k,1,n)+J_4 b_1^{(2)}(k,1,n)+J_2\Big[\mathrm{e}^{-ik/3}b_1^{(1)}(k,1,n-1)$$
$$+\mathrm{e}^{ik/3}b_1^{(1)}(k,1,n+1)+\mathrm{e}^{ik/3}b_1^{(1)}(k,2,n-1)$$
$$+\mathrm{e}^{-ik/3}b_1^{(1)}(k,2,n)\Big]=Eb_1^{(1)}(k,1,n),$$
$$\Big(6J_1+J_2+\frac{3}{2}J_3\Big)b_1^{(2)}(k,1,n)+J_4 b_1^{(1)}(k,1,n)+J_2\Big[\mathrm{e}^{-ik/3}b_1^{(2)}(k,1,n-1)$$
$$+\mathrm{e}^{ik/3}b_1^{(2)}(k,1,n+1)+\mathrm{e}^{ik/3}b_1^{(2)}(k,2,n-1)$$
$$+\mathrm{e}^{-ik/3}b_1^{(2)}(k,2,n)\Big]=Eb_1^{(2)}(k,1,n),$$
$$(4J_1+\varepsilon_0)b_1^{(3)}(k,1,n)+(J_5-J_4)(b_1^{(1)}(k,1,n)+b_1^{(2)}(k,1,n))$$
$$+J_2[\mathrm{e}^{-ik/3}b_1^{(3)}(k,1,n-1)+\mathrm{e}^{ik/3}b_1^{(3)}(k,1,n+1)+\mathrm{e}^{ik/3}b_1^{(3)}(k,2,n-1)$$
$$+\mathrm{e}^{-ik/3}b_1^{(3)}(k,2,n)]=Eb_1^{(3)}(k,1,n),$$
$$(4J_1+\varepsilon_0)b_1^{(1)}(k,m,1)+(J_5-J_4)[b_1^{(2)}(k,m,1)+b_1^{(3)}(k,m,1)]$$
$$+J_2[\mathrm{e}^{ik/3}b_1^{(1)}(k,m-1,1)+\mathrm{e}^{-ik/3}b_1^{(1)}(k,m+1,1)+\mathrm{e}^{-ik/3}b_1^{(1)}(k,m-1,2)$$

$$
\begin{aligned}
&+\mathrm{e}^{ik/3}b_1^{(1)}(k,m,2)] = Eb_1^{(1)}(k,m,1),\\
&\Big(6J_1+J_2+\frac{3}{2}J_3\Big)b_1^{(2)}(k,m,1)+J_4b_1^{(3)}(k,m,1)+J_2[\mathrm{e}^{ik/3}b_1^{(2)}(k,m-1,1)\\
&+\mathrm{e}^{-ik/3}b_1^{(2)}(k,m+1,1)+\mathrm{e}^{-ik/3}b_1^{(2)}(k,m-1,2)\\
&+\mathrm{e}^{ik/3}b_1^{(2)}(k,m,2)] = Eb_1^{(2)}(k,m,1),\\
&\Big(6J_1+J_2+\frac{3}{2}J_3\Big)b_1^{(3)}(k,m,1)+J_4b_1^{(2)}(k,m,1)+J_2[\mathrm{e}^{ik/3}b_1^{(3)}(k,m-1,1)\\
&+\mathrm{e}^{-ik/3}b_1^{(3)}(k,m+1,1)+\mathrm{e}^{-ik/3}b_1^{(3)}(k,m-1,2)\\
&+\mathrm{e}^{ik/3}b_1^{(3)}(k,m,2)] = Eb_1^{(3)}(k,m,1).
\end{aligned}
\tag{91}
$$

Introducing again the unphysical values $b_1^{(j)}(k,m,0)$ and $b_1^{(j)}(k,0,n)$ one can reduce (91) to the form (65) by producing the following system of Bethe conditions

$$
\begin{aligned}
&(\Delta_2+\Delta_1)b_1^{(1)}(k,1,n)+(\Delta_2-\Delta_1)b_1^{(2)}(k,1,n)=\mathrm{e}^{ik/3}b_1^{(1)}(k,0,n)\\
&+\mathrm{e}^{-ik/3}b_1^{(1)}(k,0,n+1),\\
&(\Delta_2+\Delta_1)b_1^{(2)}(k,1,n)+(\Delta_2-\Delta_1)b_1^{(1)}(k,1,n)=\mathrm{e}^{ik/3}b_1^{(2)}(k,0,n)\\
&+\mathrm{e}^{-ik/3}b_1^{(2)}(k,0,n+1),\\
&2\Delta_0b_1^{(3)}(k,1,n)+\tfrac{2}{3}(\Delta_0-\Delta_2)[b_1^{(1)}(k,1,n)+b_1^{(2)}(k,1,n)]\\
&=\mathrm{e}^{ik/3}b_1^{(3)}(k,0,n)+\mathrm{e}^{-ik/3}b_1^{(3)}(k,0,n+1),\\
&2\Delta_0b_1^{(1)}(k,m,1)+\tfrac{2}{3}(\Delta_0-\Delta_2)[b_1^{(2)}(k,m,1)+b_1^{(3)}(k,m,1)]\\
&=\mathrm{e}^{-ik/3}b_1^{(1)}(k,m,0)+\mathrm{e}^{ik/3}b_1^{(1)}(k,m+1,0),\\
&(\Delta_2+\Delta_1)b_1^{(2)}(k,m,1)+(\Delta_2-\Delta_1)b_1^{(3)}(k,m,1)=\mathrm{e}^{-ik/3}b_1^{(2)}(k,m,0)\\
&+\mathrm{e}^{ik/3}b_1^{(2)}(k,m+1,0),\\
&(\Delta_2+\Delta_1)b_1^{(3)}(k,m,1)+(\Delta_2-\Delta_1)b_1^{(2)}(k,m,1)=\mathrm{e}^{-ik/3}b_1^{(3)}(k,m,0)\\
&+\mathrm{e}^{ik/3}b_1^{(3)}(k,m+1,0).
\end{aligned}
\tag{92}
$$

As it was in the $S=0$ case the system (92) (as well as the three separate subsystems of the form (65) for $b_1^{(j)}(k,m,n)$) is symmetric under a duality transformation

$$
\mathcal{D}(b_1^{(j)}(k,m,n)) = \bar{b}_1^{(4-j)}(k,n,m). \tag{93}
$$

Before developing a general analysis of the system (92) we shall at once find all the cases when it may be reduced to the XXZ-type form (67).

First of all for

$$
\Delta_0=\Delta_1=\Delta_2 \tag{94}
$$

the system (92) decouples into three XXZ-type subsystems (67) and therefore is completely solvable. Namely

$$b_1^{(j)}(k, \mathbf{q}, m, n) = \alpha_j b_0(k, \mathbf{q}, m, n), \quad j = 1, 2, 3, \tag{95}$$

where $\alpha_{1,2,3}$ is a triple of arbitrary parameters. Labeling the coupling constants related to (94) by an upper index "(0)" one may readily obtain from (32)

$$J_d^{(0)} = -J_l^{(0)}, \quad J_{ll}^{(0)} = 4J_l^{(0)}, \quad \Delta_{0,1,2}^{(0)} = \frac{J_{rr}^{(0)}}{4J_l^{(0)}}, \tag{96}$$

or equivalently

$$J_4^{(0)} = J_5^{(0)} = 0, \qquad \Delta_{0,1,2}^{(0)} = 1 + \frac{3J_3^{(0)}}{2J_2^{(0)}}. \tag{97}$$

In this case according to (17) and (97) an interaction between excited triplet rungs is spin-independent. In more detail the relation between this model and XXZ chain was studied in [27].

Besides the complete separable case (94) there are also two configurations of Δ-s for which the system (92) possess a partial solution of the form (95) however with special values of the ratios α_i/α_j $(i, j = 1, 2, 3)$. Indeed substituting the ansatze (95) into the system (92) one readily makes sure that the latter may be reduced to (67) with an appropriate parameter Δ only under the following system of conditions

$$\begin{aligned}
(\Delta_1 + \Delta_2 - 2\Delta)\alpha_1 + (\Delta_2 - \Delta_1)\alpha_2 &= 0, \\
(\Delta_2 - \Delta_1)\alpha_1 + (\Delta_1 + \Delta_2 - 2\Delta)\alpha_2 &= 0, && (98) \\
(\Delta_1 + \Delta_2 - 2\Delta)\alpha_2 + (\Delta_2 - \Delta_1)\alpha_3 &= 0, \\
(\Delta_2 - \Delta_1)\alpha_2 + (\Delta_1 + \Delta_2 - 2\Delta)\alpha_3 &= 0, && (99) \\
3(\Delta_0 - \Delta)\alpha_1 + (\Delta_0 - \Delta_2)(\alpha_2 + \alpha_3) &= 0, \\
3(\Delta_0 - \Delta)\alpha_3 + (\Delta_0 - \Delta_2)(\alpha_1 + \alpha_2) &= 0. && (100)
\end{aligned}$$

It may be easily observed that a trivial solution of the subsystem (98) may be nontrivially extended as a solution of the whole system (98)-(100) only in the case (94). From the other hand a nontrivial solution of (98), namely $\alpha_1 = \alpha_2$, exists only for $\Delta = \Delta_2$. Besides for $\Delta_1 = \Delta_2 = \Delta$ the system (98) is satisfied for all $\alpha_{1,2}$. Extension of these two solutions on the subsystems (99), (100) results in

$$\alpha_1 = \alpha_2 = \alpha_3, \qquad \Delta = \Delta_0 = \Delta_2, \tag{101}$$

$$4\alpha_1 = -\alpha_2 = 4\alpha_3, \qquad \Delta = \Delta_1 = \Delta_2. \tag{102}$$

Turning to the general (XXZ-irreducible) case we suggest the Bethe ansatze,

$$b_1^{(j)}(k,m,n) = B_1^{(j)}(k,\mathbf{q})\mathrm{e}^{i(\tilde{q}_1 m+\tilde{q}_2 n)} - B_2^{(j)}(k,\mathbf{q})\mathrm{e}^{i(\tilde{q}_1 m+(\tilde{q}_1-\tilde{q}_2)n)}$$
$$+B_3^{(j)}(k,\mathbf{q})\mathrm{e}^{i(-\tilde{q}_2 m+(\tilde{q}_1-\tilde{q}_2)n)} - B_4^{(j)}(k,\mathbf{q})\mathrm{e}^{-i(\tilde{q}_2 m+\tilde{q}_1 n)}$$
$$+B_5^{(j)}(k,\mathbf{q})\mathrm{e}^{i((\tilde{q}_2-\tilde{q}_1)m-\tilde{q}_1 n)} - B_6^{(j)}(k,\mathbf{q})\mathrm{e}^{i((\tilde{q}_2-\tilde{q}_1)m+\tilde{q}_2 n)}. \tag{103}$$

Classification of states with complex $\tilde{q}_{1,2}$ has the form (82). However each $\mathcal{J}_i$ in (82) is now a set of l-s for which all $B_l^{(j)}(k,\mathbf{q}) = 0$. In the present paper we shall not study $S = 1$ and $S = 2$ three-magnon Bethe states with complex wave numbers. For the pure scattering states the duality (93) reduces on the amplitudes as follows

$$\mathcal{D}(B_l^{(j)}(k,\mathbf{q})) = \bar{B}_{l-3}^{(j)}(k,\mathbf{q}), \tag{104}$$

Substitution of (103) into (92) gives

$$\sum_{l=1}^{18} M_{il}^{(1)}(k,\mathbf{q})B_l(k,\mathbf{q}) = 0, \tag{105}$$

where the vector column $B_l(k,\mathbf{q})$ for $l = 1,...,18$ is defined as follows

$$B_{6(j-1)+m}(k,\mathbf{q}) = B_m^{(j)}(k,\mathbf{q}), \quad j = 1,2,3, \quad m = 1,...,6, \tag{106}$$

while nonzero entries of the 18×18 matrix $M^{(1)}(k,\mathbf{q})$ are the following,

$$\begin{aligned}
M_{11}^{(1)}(k,\mathbf{q}) &= -M_{16,17}^{(1)}(k,\mathbf{q}) = Z(k,\omega_5(\mathbf{q}),\Delta_0),\\
M_{22}^{(1)}(k,\mathbf{q}) &= M_{88}^{(1)}(k,\mathbf{q}) = -M_{11,12}^{(1)}(k,\mathbf{q}) = -M_{17,18}^{(1)}(k,\mathbf{q}) = Z\Big(k,\mathbf{q},\frac{\Delta_1+\Delta_2}{2}\Big),\\
M_{33}^{(1)}(k,\mathbf{q}) &= -M_{18,13}^{(1)}(k,\mathbf{q}) = Z(k,\omega_1(\mathbf{q}),\Delta_0),\\
M_{44}^{(1)}(k,\mathbf{q}) &= M_{10,10}^{(1)}(k,\mathbf{q}) = -M_{78}^{(1)}(k,\mathbf{q}) = -M_{13,14}^{(1)}(k,\mathbf{q}) = Z\Big(k,\omega_2(\mathbf{q}),\frac{\Delta_1+\Delta_2}{2}\Big),\\
M_{55}^{(1)}(k,\mathbf{q}) &= -M_{14,15}^{(1)}(k,\mathbf{q}) = Z(k,\omega_3(\mathbf{q}),\Delta_0\Big),\\
M_{66}^{(1)}(k,\mathbf{q}) &= M_{12,12}^{(1)}(k,\mathbf{q}) = -M_{9,10}^{(1)}(k,\mathbf{q}) = -M_{15,16}^{(1)}(k,\mathbf{q}) = Z\Big(k,\omega_4(\mathbf{q}),\frac{\Delta_1+\Delta_2}{2}\Big),\\
M_{77}^{(1)}(k,\mathbf{q}) &= M_{13,13}^{(1)}(k,\mathbf{q}) = -M_{45}^{(1)}(k,\mathbf{q}) = -M_{10,11}^{(1)}(k,\mathbf{q}) = Z\Big(k,\omega_5(\mathbf{q}),\frac{\Delta_1+\Delta_2}{2}\Big),\\
M_{99}^{(1)}(k,\mathbf{q}) &= M_{15,15}^{(1)}(k,\mathbf{q}) = -M_{61}^{(1)}(k,\mathbf{q}) = -M_{12,7}^{(1)}(k,\mathbf{q}) = Z\Big(k,\omega_1(\mathbf{q}),\frac{\Delta_1+\Delta_2}{2}\Big),\\
M_{11,11}^{(1)}(k,\mathbf{q}) &= M_{17,17}^{(1)}(k,\mathbf{q}) = -M_{23}^{(1)}(k,\mathbf{q}) = -M_{89}^{(1)}(k,\mathbf{q}) = Z\Big(k,\omega_3(\mathbf{q}),\frac{\Delta_1+\Delta_2}{2}\Big),\\
M_{14,14}^{(1)}(k,\mathbf{q}) &= -M_{56}^{(1)}(k,\mathbf{q}) = Z(\mathbf{q},\Delta_0),
\end{aligned}$$

$$
\begin{aligned}
M^{(1)}_{16,16}(k,\mathbf{q}) &= -M^{(1)}_{12}(k,\mathbf{q}) = Z(k,\omega_2(\mathbf{q}),\Delta_0),\\
M^{(1)}_{18,18}(k,\mathbf{q}) &= -M^{(1)}_{34}(k,\mathbf{q}) = Z(k,\omega_4(\mathbf{q}),\Delta_0),\\
M^{(1)}_{17}(k,\mathbf{q}) &= M^{(1)}_{1,13}(k,\mathbf{q}) = -M^{(1)}_{16,5}(k,\mathbf{q}) = -M^{(1)}_{16,11}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_0}{3}\mathrm{e}^{iq_2},\\
M^{(1)}_{18}(k,\mathbf{q}) &= M^{(1)}_{1,14}(k,\mathbf{q}) = -M^{(1)}_{16,4}(k,\mathbf{q}) = -M^{(1)}_{16,10}(k,\mathbf{q}) = \frac{\Delta_0-\Delta_2}{3}\mathrm{e}^{-iq_2},\\
M^{(1)}_{39}(k,\mathbf{q}) &= M^{(1)}_{3,15}(k,\mathbf{q}) = -M^{(1)}_{18,1}(k,\mathbf{q}) = -M^{(1)}_{18,7}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_0}{3}\mathrm{e}^{iq_1},\\
M^{(1)}_{3,10}(k,\mathbf{q}) &= M^{(1)}_{3,16}(k,\mathbf{q}) = -M^{(1)}_{18,6}(k,\mathbf{q}) = -M^{(1)}_{18,12}(k,\mathbf{q}) = \frac{\Delta_0-\Delta_2}{3}\mathrm{e}^{-iq_1},\\
M^{(1)}_{5,11}(k,\mathbf{q}) &= M^{(1)}_{5,17}(k,\mathbf{q}) = -M^{(1)}_{14,3}(k,\mathbf{q}) = -M^{(1)}_{14,9}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_0}{3}\mathrm{e}^{-i(q_1+q_2)},\\
M^{(1)}_{5,12}(k,\mathbf{q}) &= M^{(1)}_{5,18}(k,\mathbf{q}) = -M^{(1)}_{14,2}(k,\mathbf{q}) = -M^{(1)}_{14,8}(k,\mathbf{q}) = \frac{\Delta_0-\Delta_2}{3}\mathrm{e}^{i(q_1+q_2)},\\
M^{(1)}_{28}(k,\mathbf{q}) &= M^{(1)}_{82}(k,\mathbf{q}) = -M^{(1)}_{11,18}(k,\mathbf{q}) = -M^{(1)}_{17,12}(k,\mathbf{q}) = \frac{\Delta_1-\Delta_2}{2}\mathrm{e}^{i(q_1+q_2)},\\
M^{(1)}_{29}(k,\mathbf{q}) &= M^{(1)}_{83}(k,\mathbf{q}) = -M^{(1)}_{11,17}(k,\mathbf{q}) = -M^{(1)}_{17,11}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_1}{2}\mathrm{e}^{-i(q_1+q_2)},\\
M^{(1)}_{4,10}(k,\mathbf{q}) &= M^{(1)}_{10,4}(k,\mathbf{q}) = -M^{(1)}_{7,14}(k,\mathbf{q}) = -M^{(1)}_{13,8}(k,\mathbf{q}) = \frac{\Delta_1-\Delta_2}{2}\mathrm{e}^{-iq_2},\\
M^{(1)}_{4,11}(k,\mathbf{q}) &= M^{(1)}_{10,5}(k,\mathbf{q}) = -M^{(1)}_{7,13}(k,\mathbf{q}) = -M^{(1)}_{13,7}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_1}{2}\mathrm{e}^{iq_2},\\
M^{(1)}_{67}(k,\mathbf{q}) &= M^{(1)}_{12,1}(k,\mathbf{q}) = -M^{(1)}_{9,15}(k,\mathbf{q}) = -M^{(1)}_{15,9}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_1}{2}\mathrm{e}^{iq_1},\\
M^{(1)}_{6,12}(k,\mathbf{q}) &= M^{(1)}_{12,6}(k,\mathbf{q}) = -M^{(1)}_{9,16}(k,\mathbf{q}) = -M^{(1)}_{15,10}(k,\mathbf{q}) = \frac{\Delta_1-\Delta_2}{2}\mathrm{e}^{-iq_1}. \qquad (107)
\end{aligned}
$$

As in the case (94) for complete solvability of the $S=1$ problem it is necessary to have three independent solutions of the system (92). In the Bethe Ansatze framework (103), (105) this results in

$$\mathrm{rank}(M^{(1)}(k,\mathbf{q})) = 15, \qquad (108)$$

and therefore in

$$P^{(1)}_n(k,\mathbf{q}) = 0, \quad n = 0,1,2. \qquad (109)$$

Here $P^{(1)}_n(k,\mathbf{q})$ are coefficients of the characteristic polynom

$$|M^{(1)}(k,\mathbf{q}) - \lambda I| = \sum_{n=0}^{18} P^{(1)}_n(k,\mathbf{q})\lambda^n. \qquad (110)$$

Direct calculation based on the computer algebra system MAPLE gives

$$P^{(1)}_0(k,\mathbf{q}) = \det M^{(1)}(k,\mathbf{q}) = 0, \qquad (111)$$

therefore even in the general case $\mathrm{rank}(M^{(1)}(k,\mathbf{q})) \le 17$ and the system (105) *always* has at the minimum one solution. Its general form is represented in the Appendix A.

For the next coefficient $P_1^{(1)}(k,\mathbf{q})$ we have obtained by machinery calculations the following factorization

$$P_1^{(1)}(k,\mathbf{q}) = \frac{2}{729}\left(1-\mathrm{e}^{2iq_1}\right)^2\left(1-\mathrm{e}^{2iq_2}\right)^2\left(1-\mathrm{e}^{2i(q_1+q_2)}\right)^2\tilde{P}_1^{(1)}(k,\mathbf{q}), \qquad (112)$$

where

$$\tilde{P}_1^{(1)}(k,\mathbf{q}) = \mathrm{e}^{-i(11k+31q_1+31q_2)/3}\sum_{m,n,p\ge 0} Q_{m,n,p}(\Delta_0,\Delta_1,\Delta_2)\mathrm{e}^{i(mk+nq_1+pq_2)/3}. \qquad (113)$$

The sum in (113) contains 95052 terms (that is why $\tilde{P}_1^{(1)}(k,\mathbf{q})$ can not be represented in the format of this paper). According to (112) and (75) the condition (109) is satisfied at $n=1$ if either any two wave numbers from k_1, k_2 and k_3 coincides or at

$$\tilde{P}_1^{(1)}(k,\mathbf{q}) = 0. \qquad (114)$$

The former three cases are similar to the case $k_1 = k_2$ in the two-magnon problem studied in the Section 3. Three-magnon solutions of this type will not studied in the present paper. Turning to the Eq. (114) we shall confine ourselves by the problem of its solvability for all wave numbers. Namely we shall postulate

$$Q_{m,n,p}(\Delta_0,\Delta_1,\Delta_2) = 0, \qquad (115)$$

to be valid at all m,n,p.

Despite the system (115) depends only on a triple of unknown variables it is practically unsolvable by the MAPLE Gröbner package on a personal computer with RAM about 2 Gb. Luckily (as it may be directly checked by machinery calculation)

$$\frac{2Q_{8,4,13} - 9Q_{12,0,11} - Q_{4,8,15}}{2592\Delta_1^2\Delta_2^2\Delta_3^2} = 12(\Delta_0-\Delta_1)^2 + 15(\Delta_1-\Delta_2)^2 + 20(\Delta_2-\Delta_0)^2, \qquad (116)$$

so except (94) there are no solutions with $\Delta_0\Delta_1\Delta_2 \neq 0$.

In each of the three cases $\Delta_{0,1,2} = 0$ the reduced system (115) is essentially simpler than the initial one and may be readily solved on the personal computer. Calculations based on the Gröbner package gave four pairs of solutions. We shall represent them as sets of Δ-parameters: $\boldsymbol{\Delta} = [\Delta_0,\Delta_1,\Delta_2]$, and additionally as the corresponding sets of the coupling constants: $\mathbf{J} = [J_l, J_d, J_{rr}, J_{ll}, J_{dd}]$ and $\tilde{\mathbf{J}} = [J_2, J_3, J_4, J_5]$. Note that the parameters J_r and J_1 remain indefinite. This is rather evident because both of them

correspond to the term proportional to $\hat{Q}$ in the Hamiltonian. But according to (23) the former has no affect on the Bethe equations. Namely the solutions are the following

$$\mathbf{\Delta}^{(1,\pm)} = [\pm 1, 0, \pm 1], \quad \mathbf{J}^{(1,+)} = [1,0,0,4,0], \quad \tilde{\mathbf{J}}^{(1,\pm)} = [\pm 1, 0, 1, 1], \tag{117}$$

$$\mathbf{\Delta}^{(2,\pm)} = [0, \pm 1, 0], \quad \mathbf{J}^{(2,+)} = [1,0,-2,4,0], \quad \tilde{\mathbf{J}}^{(2,\pm)} = [\pm 1, -2, 1, 1], \tag{118}$$

$$\mathbf{\Delta}^{(3,\pm)} = [0, \pm\frac{3}{2}, \pm\frac{3}{2}], \quad \mathbf{J}^{(3,+)} = [1,0,4,0,-4], \quad \tilde{\mathbf{J}}^{(3,\pm)} = [\pm 1, 4, 0, -1], \tag{119}$$

$$\mathbf{\Delta}^{(4,\pm)} = [\mp\frac{3}{2}, 0, 0], \quad \mathbf{J}^{(4,+)} = [1,0,1,0,-4], \quad \tilde{\mathbf{J}}^{(4,\pm)} = [\pm 1, 1, 0, -1]. \tag{120}$$

It may be readily shown that the condition (27) is satisfied only for "+" solutions, while the "-" ones may be obtained from them by the symmetry (28). That is why we have omitted representations for $\mathbf{J}^{(1,2,3,4,-)}$. However they may be readily obtained from $\tilde{\mathbf{J}}^{(1,2,3,4,-)}$ using Eqs. (20).

The models related to $\mathbf{\Delta}^{1,+}$ and $\mathbf{\Delta}^{2,+}$ were first presented in [28]. Then the former one was intensively studied in [3]. Algebraic structures related to $\mathbf{\Delta}^{1,+}$ and $\mathbf{\Delta}^{3,-}$ models as well as to the model (94) with $\Delta_{0,1,2} = 1$ were presented in [23]. (However the cases (118), (120) and the general case (94) were not discussed in [23]).

As it follows from (101) and (102) all the models (117)-(120) have a XXZ-type solution (95). The remaining pair of solutions may be chosen in different ways. (In other words we do not know the best choice of basis in the two-dimensional solution subspace additional to (95)). The basises obtained by machinery calculations within MAPLE are presented in the Appendix.

6 S = 2 three-magnon sector

A $S = 2$ three magnon state related to $\mathbf{S}^z = 2$ has the following form

$$\begin{aligned} |3,2,k\rangle^a &= \sum_{m<n<p} \mathrm{e}^{ik(m+n+p)/3}[\\ & b_2^{(1)}(k, n-m, p-n)\ldots|1\rangle_m^+\ldots(|1\rangle_n^+\ldots|1\rangle_p^3 - |1\rangle_n^3\ldots|1\rangle_p^+)\ldots \\ &+ b_2^{(2)}(k, n-m, p-n)\ldots(|1\rangle_m^+\ldots|1\rangle_n^3 - |1\rangle_m^3\ldots|1\rangle_n^+)\ldots|1\rangle_p^+\ldots \end{aligned} \tag{121}$$

For $m, n > 1$ the amplitudes $b_2^{(1,2)}(k,m,n)$ separately satisfy the Schrödinger equation (65) while for $m = 1$ or $n = 1$

$$(4J_1 + \varepsilon_2)b_2^{(1)}(k,1,n) + J_2[\mathrm{e}^{-ik/3}b_2^{(1)}(k,1,n-1) + \mathrm{e}^{ik/3}b_2^{(1)}(k,1,n+1)$$

$$
\begin{gathered}
+\mathrm{e}^{ik/3}b_2^{(1)}(k,2,n-1)+\mathrm{e}^{-ik/3}b_2^{(1)}(k,2,n)]=Eb_2^{(1)}(k,1,n),\\
(4J_1+\varepsilon_1)b_2^{(2)}(k,1,n)+J_4b_2^{(1)}(k,1,n)+J_2[\mathrm{e}^{-ik/3}b_2^{(2)}(k,1,n-1)\\
+\mathrm{e}^{ik/3}b_2^{(2)}(k,1,n+1)+\mathrm{e}^{ik/3}b_2^{(2)}(k,2,n-1)\\
+\mathrm{e}^{-ik/3}b_2^{(2)}(k,2,n)]=Eb_2^{(2)}(k,1,n),\\
(4J_1+\varepsilon_2)b_2^{(2)}(k,m,1)+J_2[\mathrm{e}^{ik/3}b_2^{(2)}(k,m-1,1)+\mathrm{e}^{-ik/3}b_2^{(2)}(k,m+1,1)\\
+\mathrm{e}^{-ik/3}b_2^{(2)}(k,m-1,2)+\mathrm{e}^{ik/3}b_2^{(2)}(k,m,2)]=Eb_2^{(2)}(k,m,1),\\
(4J_1+\varepsilon_1)b_2^{(1)}(k,m,1)+J_4b_2^{(2)}(k,m,1)+J_2[\mathrm{e}^{ik/3}b_2^{(1)}(k,m-1,1)\\
+\mathrm{e}^{-ik/3}b_2^{(1)}(k,m+1,1)+\mathrm{e}^{-ik/3}b_2^{(1)}(k,m-1,2)\\
+\mathrm{e}^{ik/3}b_2^{(1)}(k,m,2)]=Eb_2^{(1)}(k,m,1).
\end{gathered}
\tag{122}
$$

Introducing again the unphysical amplitudes we obtain from (122) the corresponding system of coupled Bethe conditions

$$
\begin{gathered}
2\Delta_1b_2^{(1)}(k,m,1)+(\Delta_2-\Delta_1)b_2^{(2)}(k,m,1)=\mathrm{e}^{-ik/3}b_2^{(1)}(k,m,0)+\mathrm{e}^{ik/3}b_2^{(1)}(k,m+1,0),\\
2\Delta_2b_2^{(2)}(k,m,1)=\mathrm{e}^{-ik/3}b_2^{(2)}(k,m,0)+\mathrm{e}^{ik/3}b_2^{(2)}(k,m+1,0),\\
2\Delta_1b_2^{(2)}(k,1,n)+(\Delta_2-\Delta_1)b_2^{(1)}(k,1,n)=\mathrm{e}^{ik/3}b_2^{(2)}(k,0,n)+\mathrm{e}^{-ik/3}b_2^{(2)}(k,0,n+1),\\
2\Delta_2b_2^{(1)}(k,1,n)=\mathrm{e}^{ik/3}b_2^{(1)}(k,0,n)+\mathrm{e}^{-ik/3}b_2^{(1)}(k,0,n+1),
\end{gathered}
\tag{123}
$$

invariant under duality transformation

$$\mathcal{D}(b_2^{(j)}(k,m,n))=\bar{b}_2^{(3-j)}(k,n,m). \tag{124}$$

For

$$\Delta_1=\Delta_2, \tag{125}$$

or (according to (32))

$$J_{ll}=-4J_d, \tag{126}$$

this system decouples into a pair of the XXZ-type subsystems (67) on $b_2^{(1,2)}(k,m,n)$. In this case the general solution

$$b_2^{(j)}(k,\mathbf{q},m,n)=\beta_jb_0(k,\mathbf{q},m,n),\quad j=1,2, \tag{127}$$

depends on $\mathbf{q}$ and two arbitrary parameters $\beta_{1,2}$.

It may be readily proved that the XXZ-type solutions (127) exist only under the condition (125).

In the general case making the standard substitution

$$\begin{aligned}
b_2^{(j)}(k,m,n) = C_1^{(j)}(k,\mathbf{q})\mathrm{e}^{i(\tilde{q}_1 m+\tilde{q}_2 n)} - C_2^{(j)}(k,\mathbf{q})\mathrm{e}^{i(\tilde{q}_1 m+(\tilde{q}_1-\tilde{q}_2)n)} \\
+C_3^{(j)}(k,\mathbf{q})\mathrm{e}^{i(-\tilde{q}_2 m+(\tilde{q}_1-\tilde{q}_2)n)} - C_4^{(j)}(k,\mathbf{q})\mathrm{e}^{-i(\tilde{q}_2 m+\tilde{q}_1 n)} \\
+C_5^{(j)}(k,\mathbf{q})\mathrm{e}^{i((\tilde{q}_2-\tilde{q}_1)m-\tilde{q}_1 n)} - C_6^{(j)}(k,\mathbf{q})\mathrm{e}^{i((\tilde{q}_2-\tilde{q}_1)m+\tilde{q}_2 n)},
\end{aligned} \tag{128}$$

one results in a linear system

$$\sum_{j=1}^{12} M_{ij}^{(2)}(k,\mathbf{q})C_j(k,\mathbf{q}) = 0, \tag{129}$$

where as in (106)

$$C_{6(j-1)+m}(k,\mathbf{q}) = C_m^{(j)}(k,\mathbf{q}), \quad j=1,2, \quad m=1,...,6, \tag{130}$$

and the 12×12 matrix $M^{(2)}(k,\mathbf{q})$ has the following nonzero entries

$$\begin{aligned}
M_{11}^{(2)}(k,\mathbf{q}) &= -M_{10,11}^{(2)}(k,\mathbf{q}) = Z(k,\omega_5(\mathbf{q}),\Delta_1), \\
M_{22}^{(2)}(k,\mathbf{q}) &= -M_{11,12}^{(2)}(k,\mathbf{q}) = Z(k,\mathbf{q},\Delta_2), \\
M_{33}^{(2)}(k,\mathbf{q}) &= -M_{12,7}^{(2)}(k,\mathbf{q}) = Z(k,\omega_1(\mathbf{q}),\Delta_1), \\
M_{44}^{(2)}(k,\mathbf{q}) &= -M_{78}^{(2)}(k,\mathbf{q}) = Z(k,\omega_2(\mathbf{q}),\Delta_2), \\
M_{55}^{(2)}(k,\mathbf{q}) &= -M_{89}^{(2)}(k,\mathbf{q}) = Z(k,\omega_3(\mathbf{q}),\Delta_1), \\
M_{66}^{(2)}(k,\mathbf{q}) &= -M_{9,10}^{(2)}(k,\mathbf{q}) = Z(k,\omega_4(\mathbf{q}),\Delta_2), \\
M_{77}^{(2)}(k,\mathbf{q}) &= -M_{45}^{(2)}(k,\mathbf{q}) = Z(k,\omega_5(\mathbf{q}),\Delta_2), \\
M_{88}^{(2)}(k,\mathbf{q}) &= -M_{56}^{(2)}(k,\mathbf{q}) = Z(k,\mathbf{q},\Delta_1), \\
M_{99}^{(2)}(k,\mathbf{q}) &= -M_{61}^{(2)}(k,\mathbf{q}) = Z(k,\omega_1(\mathbf{q}),\Delta_2), \\
M_{10,10}^{(2)}(k,\mathbf{q}) &= -M_{12}^{(2)}(k,\mathbf{q}) = Z(k,\omega_2(\mathbf{q}),\Delta_1), \\
M_{11,11}^{(2)}(k,\mathbf{q}) &= -M_{23}^{(2)}(k,\mathbf{q}) = Z(k,\omega_3(\mathbf{q}),\Delta_2), \\
M_{12,12}^{(2)}(k,\mathbf{q}) &= -M_{34}^{(2)}(k,\mathbf{q}) = Z(k,\omega_4(\mathbf{q}),\Delta_1), \\
M_{17}^{(2)}(k,\mathbf{q}) &= -M_{10,5}^{(2)}(k,\mathbf{q}) = \frac{\Delta_1-\Delta_2}{2}\mathrm{e}^{iq_2}, \\
M_{18}^{(2)}(k,\mathbf{q}) &= -M_{10,4}^{(2)}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_1}{2}\mathrm{e}^{-iq_2}, \\
M_{39}^{(2)}(k,\mathbf{q}) &= -M_{12,1}^{(2)}(k,\mathbf{q}) = \frac{\Delta_1-\Delta_2}{2}\mathrm{e}^{iq_1}, \\
M_{3,10}^{(2)}(k,\mathbf{q}) &= -M_{12,6}^{(2)}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_1}{2}\mathrm{e}^{-iq_1},
\end{aligned}$$

$$M^{(2)}_{5,11}(k,\mathbf{q}) = -M^{(2)}_{83}(k,\mathbf{q}) = \frac{\Delta_1-\Delta_2}{2}\mathrm{e}^{-i(q_1+q_2)},$$
$$M^{(2)}_{5,12}(k,\mathbf{q}) = -M^{(2)}_{82}(k,\mathbf{q}) = \frac{\Delta_2-\Delta_1}{2}\mathrm{e}^{i(q_1+q_2)}. \qquad (131)$$

As in the $S=1$ case we shall concern only on the pure scattering states for which the duality (124) gives

$$\mathcal{D}(C^{(j)}_l(k,\mathbf{q})) = \bar{C}^{(j)}_{l-3}(k,\mathbf{q}). \qquad (132)$$

To be completely solvable the system (129) must posses two independent solutions. Equivalently there should be

$$\mathrm{rank}(M^{(2)}(k,\mathbf{q})) = 10. \qquad (133)$$

According to machinery calculation

$$\mathrm{det}M^{(2)}(k,\mathbf{q}) = -6(\Delta_1-\Delta_2)^2(1-\mathrm{e}^{2i(q_1+q_2)})^2(1-\mathrm{e}^{-2iq_1})^2(1-\mathrm{e}^{-2iq_2})^2Y^2(k,\mathbf{q}), \qquad (134)$$

where

$$Y(k,\mathbf{q}) = \Big[(\Delta_1-\Delta_2)^2-1\Big]\cos k + \Big[(\Delta_1+\Delta_2)^2-1\Big]\Big[\cos\frac{k-4q_1-2q_2}{3} + \cos\frac{k+2q_1+4q_2}{3} + \cos\frac{k+2q_1-2q_2}{3}\Big] - 4\Delta_1\Delta_2(\Delta_1+\Delta_2). \qquad (135)$$

A condition

$$\mathrm{det}M^{(2)}(k,\mathbf{q}) = 0, \qquad (136)$$

will be satisfied at all k, q_1 and q_2 either in the case (125) or in the four additional ones

$$\Delta_1 = \pm 1, \quad \Delta_2 = 0 \qquad (137)$$

and

$$\Delta_1 = 0, \quad \Delta_2 = \pm 1. \qquad (138)$$

Machinery calculations show that in all these cases the condition (133) is satisfied. The corresponding solutions of the system (129) are presented in the Appendix.

7 Integrability and the Reshetikhin condition

A well known alternative to the Coordinate Bethe Ansatze is the so called Algebraic Bethe Ansatze or the Inverse Scattering Method [15]-[17]. It is based on the representation of

the finite dimensional matrix H related to the local Hamiltonian density $H_{n,n+1}$ as a derivative of the corresponding R-matrix.

$$H = \frac{\partial}{\partial\lambda}\check{R}(\lambda)|_{\lambda=0}. \tag{139}$$

The latter satisfies the Yang-Baxter equation,

$$\check{R}_{12}(\lambda-\mu)\check{R}_{23}(\lambda)\check{R}_{12}(\mu) = \check{R}_{23}(\mu)\check{R}_{12}(\lambda)\check{R}_{23}(\lambda-\mu), \tag{140}$$

and the initial condition,

$$\check{R}(0) \propto I, \tag{141}$$

(where again I is an identity matrix).

From (139)-(141) follows the Reshetikhin condition [17],

$$[H_{12}+H_{23},[H_{12},H_{23}]] = K_{23} - K_{12}, \tag{142}$$

which for the Hamiltonian density (17) with $J_6 = 0$ gives the following system of equations,

$$\begin{aligned}
J_2J_4(J_1+J_3+J_5) &= 0,\\
J_2(J_4-J_5)(J_4+J_5) &= 0,\\
J_2(J_4-J_5)(2J_1+2J_3-J_4+5J_5) &= 0,\\
(J_4-J_5)(J_2^2-J_5^2+2J_4J_5) &= 0,\\
J_5(J_2^2-2J_4^2-J_5^2+2J_4J_5) &= 0,\\
J_2(J_3^2+2J_1J_3-4J_5^2+4J_4J_5) &= 0.
\end{aligned} \tag{143}$$

Taking at the first $J_2 = 0$ one gets from (143) $J_5 = 0$. This case with degenerate one-magnon dispersion is of poor physical interest and was already studied in [26]. Taking now $J_2 \neq 0$ and using (32) one can subdivide the system (143) on two subsystems,

$$\begin{aligned}
(\Delta_2-\Delta_0)(\Delta_2-\Delta_1)(3\Delta_1-2\Delta_2-\Delta_0) &= 0,\\
(\Delta_2-\Delta_0)[9(\Delta_1-\Delta_2)^2-4(\Delta_2-\Delta_0)^2+9] &= 0,\\
(3\Delta_1-\Delta_2-2\Delta_0)[9(\Delta_1-\Delta_2)^2+4(\Delta_2-\Delta_0)^2-9] &= 0,
\end{aligned} \tag{144}$$

and

$$\begin{aligned}
(\Delta_1-\Delta_2)(3\Delta_r+3\Delta_1+\Delta_2-\Delta_0) &= 0,\\
(\Delta_2-\Delta_0)(3\Delta_r+\Delta_2-3\Delta_1+5\Delta_0) &= 0,\\
3\Delta_r(3\Delta_1+\Delta_2-\Delta_0)+9\Delta_1^2-18\Delta_1\Delta_2+\Delta_2^2+4\Delta_2\Delta_0-5\Delta_0^2 &= 0,
\end{aligned} \tag{145}$$

where $\Delta_r \equiv J_r/(J_l - J_d)$.

We have separated (145) from (144) because the variable J_r related to the term proportional to $\hat{Q}$ is auxiliary and according to (23) has no affect on integrability. Nevertheless solvability of the Yang-Baxter equation (140) for a given H depends on the value of J_r. Therefore however it is necessary for the system (144), (145) to be solvable a concrete value of J_r obtained from it has no affect on integrability.

The subsystem (144) has three solutions. The first one is the solution (94) for which the subsystem (145) is also solvable. The remaining two solutions of (144) are the following,

$$\Delta_0 = \Delta_2, \quad (\Delta_1 - \Delta_2)^2 = 1, \tag{146}$$

$$\Delta_1 = \Delta_2, \quad 4(\Delta_0 - \Delta_2)^2 = 9. \tag{147}$$

A substitution of (146) into (145) shows that the latter subsystem is solvable with respect to Δ_r only for,

$$\Delta_1 \Delta_2 = 0. \tag{148}$$

Together (146) and (148) result in (117) and (118).

Analogously a substitution of (147) into (145) gives,

$$\Delta_1 \Delta_0 = 0. \tag{149}$$

Together (147) and (149) results in (119) and (120).

Corresponding to the integrable cases R-matrices were already presented in [21] within the following basis in the space $\mathbb{C}^{16}$

$$f_{3(i-1)+j} = e_i \otimes e_j, \quad f_{9+i} = |0\rangle \otimes e_j, \quad f_{12+i} = e_i \otimes |0\rangle, \quad f_{16} = |0\rangle \otimes |0\rangle. \tag{150}$$

Here $i, j = 1, 2, 3$ and $e_1 = |1\rangle^{+1}$, $e_2 = |1\rangle^{0}$ and $e_3 = |1\rangle^{-1}$.

In this basis the R-matrix corresponding to (94) has the block XXZ-type form

$$\check{R}^{(0)}(\lambda) = \begin{pmatrix} \sinh(\lambda+\eta) I_9 & 0 & 0 & 0 \\ 0 & \sinh\eta I_3 & \sinh\lambda I_3 & 0 \\ 0 & \sinh\lambda I_3 & \sinh\eta I_3 & 0 \\ 0 & 0 & 0 & \sinh(\lambda+\eta) \end{pmatrix}. \tag{151}$$

For a very special value of η it was also presented in [27].

In the cases (117) (for $J_r = 0$) and (118) ($J_r = J_l$) the matrices H are correspondingly the normal and graded $\mathbb{C}^4 \otimes \mathbb{C}^4$ permutators $\mathcal{P}_4$ and $\tilde{\mathcal{P}}_4$. (In the latter case the subspace

generated by $|0\rangle$ has the negative grading). The related R-matrices have a rather simple form,

$$\check{R}^{(1,2)}(\lambda) = \eta I_{16} + \lambda H. \tag{152}$$

Integrability of these models was first noted in [28]. The case (117) was intensively studied in [3].

The R-matrices related to (119) (for $J_r = J_l$) and (120) ($2J_r = 5J_l$) also have block forms,

$$\begin{aligned}
\check{R}^{(3)}(\lambda) &= \begin{pmatrix} r(\lambda,\eta_0) & 0 & 0 & 0 \\ 0 & \sinh\eta_0 I_3 & \sinh\lambda I_3 & 0 \\ 0 & \sinh\lambda I_3 & \sinh\eta_0 I_3 & 0 \\ 0 & 0 & 0 & \sinh(\lambda+\eta_0) \end{pmatrix}, \\
\check{R}^{(4)}(\lambda) &= \begin{pmatrix} r(\lambda,\eta_0) & 0 & 0 & 0 \\ 0 & \sinh\eta_0 I_3 & \sinh\lambda I_3 & 0 \\ 0 & \sinh\lambda I_3 & \sinh\eta_0 I_3 & 0 \\ 0 & 0 & 0 & \sinh(\eta_0-\lambda) \end{pmatrix},
\end{aligned} \tag{153}$$

where $\sinh\eta_0 = \sqrt{5}/2$ and,

$$r(\lambda,\eta_0) = \begin{pmatrix} f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f-g & 0 & g & 0 & -g & 0 & 0 \\ 0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g & 0 & f-g & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 \\ 0 & 0 & -g & 0 & g & 0 & f-g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f \end{pmatrix}, \tag{154}$$

($f = \sinh(\lambda+\eta_0)$, $g = \sinh\lambda$).

The matrix $r(\lambda,\eta_0)$ itself satisfies the Yang-Baxter equation and describes the $S=1$ biquadratic spin chain. As it was shown in [29] this R-matrix as well as its generalization (related to arbitrary η) are related to the Temperley-Lieb algebra.

8 Action of the S_3 group on the eigenspaces

As it will be shown below (see Eq. (162)) the S_3-action (87) in the q-space results in corresponding symmetry of Bethe wave functions. The latter is useful (see Appendix A) for compact representation of amplitudes.

First of all let us consider the case $S=0$ (which is analogous to $S=3$). The matrix $M^{(0)}(k,\mathbf{q})$ possess the following symmetry

$$M^{(0)}(k,\omega_j(\mathbf{q})) = J_L^{(0)}(\omega_j)M^{(0)}(k,\mathbf{q})J_R^{(0)}(\omega_j), \tag{155}$$

where the matrices $J_L^{(0)}(\omega_j)$ and $J_R^{(0)}(\omega_j)$ give left and right representations of the group S_3:

$$J_L^{(0)}(\omega_i)J_L^{(0)}(\omega_j) = J_L^{(0)}(\omega_i\cdot\omega_j), \quad J_R^{(0)}(\omega_i)J_R^{(0)}(\omega_j) = J_R^{(0)}(\omega_j\cdot\omega_i). \tag{156}$$

Explicit expressions for the matrices $J_{L,R}^{(0)}(\omega_j)$ may be obtained from Eqs. (79), (156) and the following representations for generators

$$J_L^{(0)}(\omega_1) = \begin{pmatrix} 1 & \mathbb{O}_{1,5} \\ \mathbb{O}_{5,1} & \tilde{I}_5 \end{pmatrix}, \qquad J_L^{(0)}(\omega_5) = \begin{pmatrix} \tilde{I}_5 & \mathbb{O}_{5,1} \\ \mathbb{O}_{1,5} & 1 \end{pmatrix},$$

$$J_R^{(0)}(\omega_1) = -\begin{pmatrix} \tilde{I}_2 & \mathbb{O}_{2,4} \\ \mathbb{O}_{4,2} & \tilde{I}_4 \end{pmatrix}, \qquad J_R^{(0)}(\omega_5) = -\tilde{I}_6. \tag{157}$$

Here by $\mathbb{O}_{m,n}$ we denote a $m\times n$ matrix with all zero entries while by $\tilde{I}_n$ a $n\times n$ matrix with units in the second diagonal (and all other entries equal to zero).

Similar relations

$$M^{(1,2)}(k,\omega_j(\mathbf{q})) = J_L^{(1,2)}(\omega_j)M^{(1,2)}(k,\mathbf{q})J_R^{(1,2)}(\omega_j), \tag{158}$$

with

$$J_{L,R}^{(1)} = I_3\otimes J_{L,R}^{(0)}, \quad J_{L,R}^{(2)} = I_2\otimes J_{L,R}^{(0)}, \tag{159}$$

are also valid for $M^{(1,2)}(k,\mathbf{q})$ given by (107) and (131).

The symmetry (158) allows to produce new solutions of the Eqs. (105) or (129) from the known one (for Eq. (84) the result is trivial). Indeed if

$$M^{(n)}(k,\mathbf{q})F^{(n)}(k,\mathbf{q}) = 0, \tag{160}$$

for some vector $F^{(n)}(k,\mathbf{q})$ ($\dim(F^{(1)}(k,\mathbf{q})) = 18$, $\dim(F^{(2)}(k,\mathbf{q})) = 12$) then according to (158)

$$M^{(n)}(k,\mathbf{q})J_R^{(n)}(\omega_j)F^{(n)}(k,\omega_j(\mathbf{q})) = 0. \tag{161}$$

In other words we have obtained the following action of the group S_3 on the eigenspaces

$$\omega_j(F^{(n)})(k,\mathbf{q}) = J_R^{(n)}(\omega_j)F^{(n)}(k,\omega_j(\mathbf{q})). \tag{162}$$

Here $\omega_j(F^{(n)})$ is vector related to the new solution (which in fact may coincide with the present one).

9 Summary

In the present paper we analyzed two- and three-magnon problems for a rung-dimerized spin ladder. It was shown that the Bethe form of the two-magnon solution may be obtained in a straightforward manner from the corresponding Shrödinger equation.

The three-magnon problem was first analyzed in general outlook in all sectors of total spin $S = 0, 1, 2, 3$. It was shown that at all S the reduced to the center of mass frame Shrödinger equation is invariant under the corresponding duality transformation while the Fourier substitution (70) naturally results in Bethe form of wave function.

Applicability of the Bethe Ansatze for the three-magnon problem was analyzed separately in all sectors of total spin. It was shown that for $S = 0$ and $S = 3$ the problem is always solvable and the corresponding solution has form typical to the XXZ model. The sector $S = 1$ is completely solvable in the five cases (94) and (117)-(120). Nevertheless a special partial solution (see Appendix A) exist for all values of the coupling constants. The sector $S = 2$ is solvable under one of the conditions (125), (137) or (138). Explicit expressions for the solutions are presented in the Appendix.

The result was compared with the previous consideration based on an analysis of solvability of the Yang-Baxter equation. It was shown that the three-magnon problem for a Hamiltonian $\hat{H}$ is completely solvable within the Coordinate Bethe Ansatze if and only if the corresponding R-matrix exist for some Hamiltonian in the orbit $\hat{H} + \alpha\hat{Q}$ (α is real).

Finitely it is shown that the S_3-symmetry of the Bethe Ansatze equations results in the action (162) of the group S_3 on the space of Bethe vectors.

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A Partial solution in the $S = 1$ sector

An explicit form of the special partial solution of Eq. (105) obtained by MAPLE is rather complicated. For example the expressions for $B_j(k,\mathbf{q})$ at $j = 1,..,6$ and $j = 13,..,18$

contain 1106 terms while the expression for $B_j(k,\mathbf{q})$ at $j=7,..,12$ contain 1090.

Since this solution is in general a single one it must be S_3-symmetric and auto- (or anti-auto) dual. It may be readily proved that these symmetry properties allow to obtain all components from $B_1(k,\mathbf{q})$ and $B_7(k,\mathbf{q})$ using Eqs. (104) and (162). Below we give representations for these two components.

First of all $B_1(k,\mathbf{q})$ possess the following decomposition

$$B_1(k,\mathbf{q}) = B_1^{(s)}(k,\mathbf{q}) + B_1^{(a)}(k,\mathbf{q}), \tag{163}$$

where the term $B_1^{(s)}(k,\mathbf{q})$ is symmetric under the transposition,

$$k \to -k, \quad q_1 \leftrightarrow q_2, \tag{164}$$

while $B_1^{(a)}(k,\mathbf{q})$ is antisymmetric.

For $B_1^{(s)}(k,\mathbf{q})$ we have the following representation

$$\begin{aligned} B_1^{(s)}(k,\mathbf{q}) &= \frac{45}{2}F(\Delta_1,\Delta_2,\Delta_0) + \frac{45}{2}F(\Delta_2,\Delta_0,\Delta_1) + \frac{9}{2}F(\Delta_0,\Delta_1,\Delta_2) \\ &+ u_1u_2u_3\Big[W_1u_1u_2u_3 + W_2\frac{u_1u_2}{z_1z_2} + W_3\Big(\frac{u_1z_2}{z_1^2} + \frac{u_2z_1}{z_2^2}\Big)u_3 \\ &+ W_4\frac{u_3}{z_1z_2} + W_5\Big(\frac{u_1}{z_1^3} + \frac{u_2}{z_2^3}\Big) + \frac{W_6}{z_1^2z_2^2}\Big], \end{aligned} \tag{165}$$

where

$$\begin{aligned} F(\Delta,\Delta',\Delta'') &= (\Delta'-\Delta)Z(k,\mathbf{q},\Delta'')Z(k,\omega_2(\mathbf{q}),\Delta'')Z(k,\omega_4(\mathbf{q}),\Delta'') \\ &\cdot \quad [u_1u_2u_3 + \Delta\Delta'\cos k + \Delta\Delta'(\Delta+\Delta')]. \end{aligned} \tag{166}$$

The parameters

$$\begin{aligned} u_1 &= \cos\Big(\frac{k}{3}+\frac{\tilde q_1}{2}\Big), \quad u_2 = \cos\Big(\frac{k}{3}-\frac{\tilde q_2}{2}\Big), \\ u_3 &= \cos\Big(\frac{k}{3}+\frac{\tilde q_2-\tilde q_1}{2}\Big), \quad z_j = \mathrm{e}^{i\tilde q_j/2}, \end{aligned} \tag{167}$$

have more simple form being expressed from $\tilde q_{1,2}$.

The coefficients W_j for $j=1,2,3$ are the following

$$\begin{aligned} W_1 &= 27\Delta_1^3 - 5\Delta_2^3 + 8\Delta_0^3 + 45\Delta_1^2\Delta_2 - 75\Delta_1\Delta_2^2 - 90\Delta_2^2\Delta_0 \\ &+ 60\Delta_2\Delta_0^2 - 18\Delta_1^2\Delta_0 - 12\Delta_1\Delta_0^2 + 60\Delta_1\Delta_2\Delta_0, \\ W_2 &= 18\Delta_1^3\Delta_2 - 10\Delta_1\Delta_2^3 - 45\Delta_1^3\Delta_0 - 50\Delta_1\Delta_0^3 + 15\Delta_2^3\Delta_0 + 42\Delta_2\Delta_0^3 \\ &+ 30(3\Delta_1^2 - \Delta_2^2)\Delta_0^2 + 3\Delta_1\Delta_2\Delta_0(65\Delta_2 - 39\Delta_1 - 36\Delta_0), \\ W_3 &= W_2 + 15(\Delta_1-\Delta_2)(\Delta_1-\Delta_0)(\Delta_2-\Delta_0)(4\Delta_0 - 3\Delta_1 - \Delta_2). \end{aligned} \tag{168}$$

For $j = 4, 5, 6$ they may be obtained from (168) by the following formulas (observed purely empirically)

$$W_4 = \frac{\varphi(W_2)}{\Delta_1\Delta_2\Delta_0}, \quad W_5 = \frac{\varphi(W_3)}{\Delta_1\Delta_2\Delta_0}, \quad W_6 = \varphi(W_1), \tag{169}$$

where the homomorphism φ is defined as follows,

$$\varphi(\Delta_j) = \frac{\Delta_1\Delta_2\Delta_0}{\Delta_j}. \tag{170}$$

For $B_1^{(a)}(k, \mathbf{q})$ we found the following representation

$$B_1^{(a)}(k, \mathbf{q}) = \frac{15}{2}(\Delta_1 - \Delta_2)(\Delta_1 - \Delta_0)(\Delta_2 - \Delta_0)u_3\tilde{B}_1^{(a)}(k, \mathbf{q}), \tag{171}$$

where

$$\begin{aligned} \tilde{B}_1^{(a)}(k, \mathbf{q}) &= u_1u_2\Big[(3 - 4\Delta_1\Delta_2 - 6\Delta_0\Delta_2 - 2\Delta_0\Delta_1)\Big(\frac{u_1}{z_1^3} - \frac{u_2}{z_2^3}\Big) \\ &+ 3i\sin k + 3i(1 + 4(\Delta_0\Delta_1 + \Delta_1\Delta_2 + \Delta_0\Delta_2))\frac{v_3}{z_1z_2} \\ &- 2i(2\Delta_0 + 3\Delta_1 + \Delta_2)\Big(\frac{u_1z_2}{z_1^2} + \frac{u_2z_1}{z_2^2}\Big)v_3 \\ &- 2i(\Delta_0 + 2\Delta_2)\frac{u_1v_2 + u_2v_1}{z_1z_2}\Big] - 6i\Delta_0\Delta_1\Delta_2\frac{u_1v_2 + u_2v_1}{z_1^2z_2^2}, \end{aligned} \tag{172}$$

and

$$v_1 = \sin\Big(\frac{k}{3} + \frac{\tilde{q}_1}{2}\Big), \quad v_2 = \sin\Big(\frac{k}{3} - \frac{\tilde{q}_2}{2}\Big), \quad v_3 = \sin\Big(\frac{k}{3} + \frac{\tilde{q}_2 - \tilde{q}_1}{2}\Big). \tag{173}$$

Representation of $B_7(k, \mathbf{q})$ is similar to (165)

$$\begin{aligned} B_7(k, \mathbf{q}) &= 45F(\Delta_0, \Delta_2, \Delta_1) + 27F(\Delta_0, \Delta_1, \Delta_2) \\ &+ u_1u_2u_3\Big[\tilde{W}_1u_1u_2u_3 + \tilde{W}_2\frac{u_1u_2}{z_1z_2} + \tilde{W}_3\Big(\frac{u_1z_2}{z_1^2} + \frac{u_2z_1}{z_2^2}\Big)u_3 \\ &+ \tilde{W}_4\frac{u_3}{z_1z_2} + \tilde{W}_5\Big(\frac{u_1}{z_1^3} + \frac{u_2}{z_2^3}\Big) + \frac{\tilde{W}_6}{z_1^2z_2^2}\Big], \end{aligned} \tag{174}$$

where

$$\begin{aligned} \tilde{W}_1 &= 27\Delta_1^3 + 5\Delta_2^3 - 32\Delta_0^3 + 45\Delta_1\Delta_2(\Delta_2 - \Delta_1) \\ &+ 72\Delta_1\Delta_0(\Delta_1 - \Delta_0) + 120\Delta_2\Delta_0(\Delta_2 - \Delta_0), \\ \tilde{W}_2 &= 45\Delta_1^3\Delta_0 - 72\Delta_1^3\Delta_2 - 80\Delta_2^3\Delta_1 + 75\Delta_2^3\Delta_0 + 20\Delta_0^3\Delta_1 + 12\Delta_0^3\Delta_2 \\ &+ 60(2\Delta_1^2\Delta_2^2 - \Delta_1^2\Delta_0^2 - \Delta_2^2\Delta_0^2) + 3\Delta_1\Delta_2\Delta_0(104\Delta_0 - 29\Delta_1 - 75\Delta_2), \\ \tilde{W}_3 &= W_2 + 90(\Delta_1 - \Delta_2)^2(\Delta_1 - \Delta_0)(\Delta_2 - \Delta_0). \end{aligned} \tag{175}$$

Again the parameters $\tilde{W}_{4,5,6}$ may be obtained from $\tilde{W}_{1,2,3}$ according to (169) and (170).

B Additional $S = 1$ solutions

We shall use here the following notations

$$
\begin{aligned}
m_1(k,\mathbf{q},\Delta) &= Z(k,\omega_5(\mathbf{q}),\Delta), \quad m_2(k,\mathbf{q},\Delta) = Z(k,\mathbf{q},\Delta),\\
m_3(k,\mathbf{q},\Delta) &= = Z(k,\omega_1(\mathbf{q}),\Delta), \quad m_4(k,\mathbf{q},\Delta) = Z(k,\omega_2(\mathbf{q}),\Delta),\\
m_5(k,\mathbf{q},\Delta) &= Z(k,\omega_3(\mathbf{q}),\Delta), \quad m_6(k,\mathbf{q},\Delta) = Z(k,\omega_4(\mathbf{q}),\Delta),
\end{aligned} \tag{176}
$$

(for definition of $Z(k,\mathbf{q},\Delta)$ and $\omega_j(\mathbf{q})$ see (86) and (87)).

For $\Delta_0 = \Delta_2 = 1$, $\Delta_1 = 0$ the space of additional to (95), (101) solutions is generated by the vector

$$
\begin{aligned}
B_1^{(1)}(k,\mathbf{q}) &= 2m_4(k,\mathbf{q},1)m_6(k,\mathbf{q},1)\sin\frac{k+2q_1+4q_2}{6}\sin\frac{k-4q_1-2q_2}{6},\\
B_2^{(1)}(k,\mathbf{q}) &= 2m_1(k,\mathbf{q},1)m_6(k,\mathbf{q},1)\sin\frac{k+2q_1+4q_2}{6}\sin\frac{k-4q_1-2q_2}{6},\\
B_3^{(1)}(k,\mathbf{q}) &= -m_1(k,\mathbf{q},1)m_2(k,\mathbf{q},1)m_6(k,\mathbf{q},1),\\
B_4^{(1)}(k,\mathbf{q}) &= -m_1(k,\mathbf{q},1)m_2(k,\mathbf{q},1)m_3(k,\mathbf{q},1),\\
B_5^{(1)}(k,\mathbf{q}) &= 2m_2(k,\mathbf{q},1)m_3(k,\mathbf{q},1)\sin\frac{k+2q_1+4q_2}{6}\sin\frac{k+2q_1-2q_2}{6},\\
B_6^{(1)}(k,\mathbf{q}) &= 2m_3(k,\mathbf{q},1)m_5(k,\mathbf{q},1)\sin\frac{k+2q_1+4q_2}{6}\sin\frac{k+2q_1-2q_2}{6},\\
B_1^{(2)}(k,\mathbf{q}) &= 2i\sin(q_1+q_2)\sin\frac{k+2q_1+4q_2}{6}\sin\frac{k+2q_1-2q_2}{6}m_6(k,\mathbf{q},1),\\
B_2^{(2)}(k,\mathbf{q}) &= -im_1(k,\mathbf{q},1)m_6(k,\mathbf{q},1)\sin(q_1+q_2),\\
B_5^{(2)}(k,\mathbf{q}) &= -im_2(k,\mathbf{q},1)m_3(k,\mathbf{q},1)\sin q_2,\\
B_6^{(2)}(k,\mathbf{q}) &= 2im_3(k,\mathbf{q},1)\sin q_2\sin\frac{k+2q_1+4q_2}{6}\sin\frac{k-4q_1-2q_2}{6},\\
B_1^{(3)}(k,\mathbf{q}) &= m_6(k,\mathbf{q},1)\sin q_2\sin(q_1+q_2),\\
B_6^{(3)}(k,\mathbf{q}) &= m_3(k,\mathbf{q},1)\sin q_2\sin(q_1+q_2),\\
B_l^{(j)}(k,\mathbf{q}) &= 0, \quad (j,l) = (2,3-4),\ (3,2-5),
\end{aligned} \tag{177}
$$

and its dual.

For $\Delta_0 = \Delta_2 = 0$, $\Delta_1 = 1$ the space of additional to (95), (101) solutions is generated by the vector

$$
\begin{aligned}
B_1^{(1)}(k,\mathbf{q}) &= B_2^{(1)}(k,\mathbf{q}) = m_5(k,\mathbf{q},1)m_6(k,\mathbf{q},1),\\
B_3^{(1)}(k,\mathbf{q}) &= B_4^{(1)}(k,\mathbf{q}) = -2m_6(k,\mathbf{q},1)\sin\frac{k+2q_1+4q_2}{6}\sin\frac{k-4q_1-2q_2}{6},
\end{aligned}
$$

$$
\begin{aligned}
B_5^{(1)}(k,\mathbf{q}) &= B_6^{(1)}(k,\mathbf{q}) = -2m_5(k,\mathbf{q},1)\sin\frac{k+2q_1-2q_2}{6}\sin\frac{k-4q_1-2q_2}{6},\\
B_3^{(2)}(k,\mathbf{q}) &= im_6(k,\mathbf{q},1)\sin(q_1+q_2),\\
B_4^{(2)}(k,\mathbf{q}) &= -2i\sin(q_1+q_2)\sin\frac{k+2q_1-2q_2}{6}\sin\frac{k-4q_1-2q_2}{6}\\
B_5^{(2)}(k,\mathbf{q}) &= -2i\sin q_1\sin\frac{k+2q_1+4q_2}{6}\sin\frac{k-4q_1-2q_2}{6},\\
B_6^{(2)}(k,\mathbf{q}) &= im_5(k,\mathbf{q},1)\sin q_1,\\
B_4^{(3)}(k,\mathbf{q}) &= B_5^{(3)}(k,\mathbf{q}) = -\sin q_1\sin(q_1+q_2),\\
B_l^{(j)}(k,\mathbf{q}) &= 0,\quad (j,l)=(2,1-2),\ (3,1-3),\ (3,6),
\end{aligned}
\tag{178}
$$

and its dual.

For $\Delta_1=\Delta_2=3/2$, $\Delta_0=0$ the space of additional to (95), (102) solutions is generated by the vector

$$
\begin{aligned}
B_2^{(1)}(k,\mathbf{q}) &= -2m_5(k,\mathbf{q},3/2)\sin q_1\sin q_2,\\
B_3^{(1)}(k,\mathbf{q}) &= -2m_2(k,\mathbf{q},3/2)\sin q_1\sin q_2,\\
B_1^{(2)}(k,\mathbf{q}) &= m_4(k,\mathbf{q},3/2)m_5(k,\mathbf{q},3/2)m_6(k,\mathbf{q},3/2),\\
B_2^{(2)}(k,\mathbf{q}) &= m_1(k,\mathbf{q},3/2)m_5(k,\mathbf{q},3/2)m_6(k,\mathbf{q},3/2),\\
B_3^{(2)}(k,\mathbf{q}) &= m_1(k,\mathbf{q},3/2)m_2(k,\mathbf{q},3/2)m_6(k,\mathbf{q},3/2),\\
B_4^{(2)}(k,\mathbf{q}) &= m_1(k,\mathbf{q},3/2)m_2(k,\mathbf{q},3/2)m_3(k,\mathbf{q},3/2),\\
B_5^{(2)}(k,\mathbf{q}) &= -B_5^{(3)}(k,\mathbf{q}) = m_2(k,\mathbf{q},3/2)m_3(k,\mathbf{q},3/2)m_4(k,\mathbf{q},3/2),\\
B_6^{(2)}(k,\mathbf{q}) &= -B_6^{(3)}(k,\mathbf{q}) = m_3(k,\mathbf{q},3/2)m_4(k,\mathbf{q},3/2)m_5(k,\mathbf{q},3/2),\\
B_1^{(3)}(k,\mathbf{q}) &= -m_4(k,\mathbf{q},3/2)m_5(k,\mathbf{q},3/2)\left[\cos\left(\frac{k-q_1-2q_2}{3}\right)-\cos q_1-\frac{\mathrm{e}^{iq_1}}{2}\right],\\
B_2^{(3)}(k,\mathbf{q}) &= -m_1(k,\mathbf{q},3/2)m_5(k,\mathbf{q},3/2)\left[\cos\left(\frac{k-q_1-2q_2}{3}\right)-\cos q_1-\frac{\mathrm{e}^{iq_1}}{2}\right],\\
B_3^{(3)}(k,\mathbf{q}) &= -m_2(k,\mathbf{q},3/2)m_6(k,\mathbf{q},3/2)\left[\cos\left(\frac{k+2q_1+q_2}{3}\right)-\cos q_2-\frac{\mathrm{e}^{-iq_2}}{2}\right],\\
B_4^{(3)}(k,\mathbf{q}) &= -m_2(k,\mathbf{q},3/2)m_3(k,\mathbf{q},3/2)\left[\cos\left(\frac{k+2q_1+q_2}{3}\right)-\cos q_2-\frac{\mathrm{e}^{-iq_2}}{2}\right],\\
B_j(k,\mathbf{q}) &= 0,\quad (j,l)=(1,1),\ (1,4-6),
\end{aligned}
\tag{179}
$$

and its dual.

For $\Delta_0=-3/2$, $\Delta_1=\Delta_2=0$ the space of additional to (95), (102) solutions is generated by the vector

$$
B_1^{(1)}(k,\mathbf{q}) = B_6^{(1)}(k,\mathbf{q}) = -im_5(k,\mathbf{q},-3/2)m_6(k,\mathbf{q},-3/2)\sin q_2,
$$

$$
\begin{aligned}
B_4^{(1)}(k,\mathbf{q}) &= B_5^{(1)}(k,\mathbf{q}) = im_1(k,\mathbf{q},-3/2)\sin q_1 \\
&\cdot \left(\cos\frac{k+q_2-q_1}{3} + \cos(q_1+q_2) + \frac{\mathrm{e}^{-i(q_1+q_2)}}{2}\right), \\
B_l^{(2)}(k,\mathbf{q}) &= m_1(k,\mathbf{q},-3/2)m_5(k,\mathbf{q},-3/2)m_6(k,\mathbf{q},-3/2), \quad l=1,2,3,4,5,6, \\
B_3^{(3)}(k,\mathbf{q}) &= B_4^{(3)}(k,\mathbf{q}) = im_1(k,\mathbf{q},-3/2)m_6(k,\mathbf{q},-3/2)\sin(q_1+q_2), \\
B_5^{(3)}(k,\mathbf{q}) &= B_6^{(3)}(k,\mathbf{q}) = im_5(k,\mathbf{q},-3/2)\sin q_1 \\
&\cdot \left(\cos\frac{k+2q_1+q_2}{3} + \cos q_2 + \frac{\mathrm{e}^{iq_2}}{2}\right), \\
B_l^{(j)}(k,\mathbf{q}) &= 0, \quad (j,l) = (1,2-3),\ (3,1-2),
\end{aligned} \tag{180}
$$

and its dual.

C $S=2$ solutions

For $\Delta_1 = \pm 1$, $\Delta_2 = 0$ the space of solutions is spanned on

$$
\begin{aligned}
C_1^{(1)}(k,\mathbf{q}) &= C_6^{(1)}(k,\mathbf{q}) = -im_5(k,\mathbf{q})m_6(k,\mathbf{q},\pm1)\sin q_2, \\
C_2^{(1)}(k,\mathbf{q}) &= C_3^{(1)}(k,\mathbf{q}) = 0, \\
C_4^{(1)}(k,\mathbf{q}) &= C_5^{(1)}(k,\mathbf{q}) = im_1(k,\mathbf{q},\pm1)m_2(k,\mathbf{q},\pm1)\sin q_1, \\
C_1^{(2)}(k,\mathbf{q}) &= C_2^{(2)}(k,\mathbf{q}) = \pm m_1(k,\mathbf{q},\pm1)m_5(k,\mathbf{q},\pm1)m_6(k,\mathbf{q},\pm1), \\
C_3^{(2)}(k,\mathbf{q}) &= C_4^{(2)}(k,\mathbf{q}) = \pm m_1(k,\mathbf{q},\pm1)m_2(k,\mathbf{q},\pm1)m_6(k,\mathbf{q},\pm1), \\
C_5^{(2)}(k,\mathbf{q}) &= C_6^{(2)}(k,\mathbf{q}) = \pm m_2(k,\mathbf{q},\pm1)m_5(k,\mathbf{q},\pm1)\Big(\mathrm{e}^{i(k+2q_1-2q_2)/6} \\
&\mp \ \mathrm{e}^{-i(k+2q_1-2q_2)/6}\Big),
\end{aligned} \tag{181}
$$

and its dual.

For $\Delta_1 = 0$, $\Delta_2 = \pm 1$ the space of solutions is spanned on

$$
\begin{aligned}
C_1^{(1)}(k,\mathbf{q}) &= C_6^{(1)}(k,\mathbf{q}) = 0, \\
C_2^{(1)}(k,\mathbf{q}) &= im_5(k,\mathbf{q},\pm1)m_6(k,\mathbf{q},\pm1)\sin q_2, \\
C_3^{(1)}(k,\mathbf{q}) &= im_2(k,\mathbf{q},\pm1)m_6(k,\mathbf{q},\pm1)\sin q_2, \\
C_4^{(1)}(k,\mathbf{q}) &= im_1(k,\mathbf{q},\pm1)m_6(k,\mathbf{q},\pm1)\sin(q_1+q_2), \\
C_5^{(1)}(k,\mathbf{q}) &= im_4(k,\mathbf{q},\pm1)m_6(k,\mathbf{q},\pm1)\sin(q_1+q_2), \\
C_1^{(2)}(k,\mathbf{q}) &= C_6^{(2)}(k,\mathbf{q},\pm1) = \mp m_4(k,\mathbf{q},\pm1)m_5(k,\mathbf{q},\pm1)m_6(k,\mathbf{q},\pm1), \\
C_2^{(2)}(k,\mathbf{q}) &= \mp m_1(k,\mathbf{q},\pm1)m_5(k,\mathbf{q},\pm1)m_6(k,\mathbf{q},\pm1),
\end{aligned}
$$

$$
\begin{aligned}
C_3^{(2)}(k,\mathbf{q}) &= \mp m_6^2(k,\mathbf{q},\pm 1)\Big(\mathrm{e}^{i(k+2q_1+4q_2)/6} \\
&\mp \mathrm{e}^{-i(k+2q_1+4q_2)/6}\Big), \\
C_4^{(2)}(k,\mathbf{q}) &= \mp m_3(k,\mathbf{q},\pm 1)m_6(k,\mathbf{q},\pm 1)\Big(\mathrm{e}^{i(k+2q_1+4q_2)/6} \\
&\mp \mathrm{e}^{-i(k+2q_1+4q_2)/6}\Big), \\
C_5^{(2)}(k,\mathbf{q}) &= \mp m_2(k,\mathbf{q},\pm 1)m_4(k,\mathbf{q},\pm 1)m_6(k,\mathbf{q},\pm 1),
\end{aligned} \tag{182}
$$

and its dual.

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